A

TREATISE ON STATICS

BY

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PREFACE

TO THE FIFTH EDITION.

In December, 1913, when the preparations for a new edition became necessary, I conferred with Professor Minchin, and, on behalf of the other Delegates of the Press, suggested (1) that certain portions should be omitted (in the hope that they might form the basis of a separate work); (2) that some account of his recent researches in Spherical Harmonics should be given; (3) that a substantial number of examples should be added. The author determined what Chapters and Articles should be left out, and readily agreed to the other proposals. Unfortunately the state of his health prevented him from even beginning the task of revision, and after his lamented death in March, 1914, it fell to other hands to attempt that which he would have performed with his recognized ability.

I have appended a large number of examples, the large majority of which have been set in Oxford Examinations. Some, with the permission of the Controller of His Majesty's Stationery Office, have been taken from Civil Service Examination Papers, and a few are based on original memoirs. No attempt has been made to secure uniformity of notation in these exercises, but hints for the solution of some have been supplied.

For convenience of reference the old numbering of the Articles has been retained.

I wish that more ample leisure had enabled me to render this edition a more worthy return for numerous kindnesses.

H. T. G.

PREFACE

TO THE SECOND VOLUME.

THE subject-matter of this second volume differing very greatly from that of the first, a few words with regard to the manner in which I have treated it seem to be necessary.

The reader will observe that in the Chapter dealing with Virtual Work, I have ventured to reject the term 'Generalized Component of Force,' and to replace it by the term 'Work Coefficient,' the former term being, to my mind, open to the objection of conveying an erroneous idea with regard to the nature of the thing defined.

In the Chapter on Attractions the great object which I have constantly kept in view has been the fixing of a *definiteness of idea* in the mind of the student with regard to the various physical magnitudes which are represented by symbols in our equations. To this end, I have explicitly adopted the C. G. S. system, and I have introduced a sufficient number of numerical illustrations in which Forces and Potentials are definitely presented as so many Dynes and so many Ergs per gramme. The C. G. S. system stands pre-eminent for its extreme simplicity; and when once the student of Mathematical Physics learns how to make a real working use of its units—to recognize, without mental effort and as a mere matter of course, that his symbol, ρ, for volume-density always means so many grammes per cubic centimetre; that his symbol, X, for force-intensity means so many dynes per gramme; and so on—he will never experience any difficulty in altering the values of fundamental numerical constants to suit the units of mass, time, and length which are adopted in any other system. In the calculation of Attractions—and especially in the domains of Electricity and Magnetism—the ever present notion of a *concrete reality* corresponding to every algebraic symbol is of immense importance. Indeed, without this definiteness of idea, no knowledge of the slightest value can exist.

STATICS.

CHAPTER XIII.

NON-COPLANAR FORCES.

ARTICLE 198.] **Resultant of any Number of Forces applied to a Material Particle.** Let a force P, represented in magnitude and direction by OO' (Fig. 228), act on a particle at O; let Ox, Oy, and Oz be any three rectangular axes drawn through O; and let the angles, $O'Ox$, $O'Oy$, and $O'Oz$, which the direction of P makes with the axes of reference be denoted by α, β, and γ, respectively. From O' let fall perpendiculars, $O'F$, $O'H$, $O'D$, on the planes yz, zx, and xy, respectively, and let the parallelepiped be completed as in the figure. Then the force OO' may be replaced by the forces OD and OC, by the parallelogram of forces; and OD can again be replaced by OA and OB. Hence the force P is equivalent to the three forces

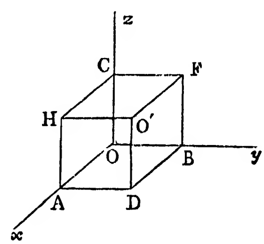

Fig. 228.

$$P \cos \alpha \text{ along } Ox,$$
$$P \cos \beta \quad ,, \quad Oy,$$
and
$$P \cos \gamma \quad ,, \quad Oz.$$

The converse proposition is also evidently true—namely, that any three forces, OA, OB, OC, along Ox, Oy, Oz (whether these are mutually rectangular directions or not), give a resultant represented in magnitude and direction by the diagonal, OO', of the parallelepiped determined by the forces.

If several forces, $P_1, P_2, \dots P_n$, act at O and make angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $\dots (\alpha_n, \beta_n, \gamma_n)$ with the axes, let each of them be replaced by its three components along Ox, Oy, Oz;

and if ΣX, ΣY, ΣZ denote the sums of the components along the axes, we shall have

$$\left.\begin{aligned}\Sigma X &= P_1 \cos\alpha_1 + P_2 \cos\alpha_2 + \ldots + P_n \cos\alpha_n,\\ \Sigma Y &= P_1 \cos\beta_1 + P_2 \cos\beta_2 + \ldots + P_n \cos\beta_n,\\ \Sigma Z &= P_1 \cos\gamma_1 + P_2 \cos\gamma_2 + \ldots + P_n \cos\gamma_n,\end{aligned}\right\} \qquad (1)$$

and the whole system of forces will be replaced by the three forces, ΣX, ΣY, and ΣZ along the axes of x, y, and z. But the resultant of three forces in these directions is the diagonal of the parallelepiped determined by them. Hence, R being the magnitude of this resultant,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}, \qquad (2)$$

and if θ, ϕ, ψ, are the direction-angles of R,

$$\cos\theta = \frac{\Sigma X}{R}, \quad \cos\phi = \frac{\Sigma Y}{R}, \quad \cos\psi = \frac{\Sigma Z}{R}. \qquad (3)$$

199.] **Graphic Representations of the Resultant.** Since the resultant of any two forces, OA and OB, acting at O is obtained by drawing from A a line, Ab, parallel and equal to OB, and joining O to b, it follows that if a particle is acted on by n forces, OA_1, OA_2, OA_3, ... OA_n, the resultant is obtained in magnitude and direction by drawing $A_1 a_2$ parallel and equal to OA_2, $a_2 a_3$ parallel and equal to OA_3, ... $a_{n-1} a_n$ parallel and equal to OA_n, and joining O to a_n; or, in other words, the side Oa_n which closes the polygon $OA_1 a_2 a_3 \ldots a_n$ represents the resultant in magnitude and direction. When the sides of the polygon are not all coplanar, the figure is called a *gauche polygon*. Thus the second graphic representation of the resultant of a system of coplanar forces, which has been given in p. 23, vol. i, is equally applicable to non-coplanar forces. Hence, of course, it follows that a particle acted on by any set of forces which are parallel and proportional to the sides of a gauche polygon taken in order is at rest.

Again, since by the parallelogram of forces the resultant of OA_1 and OA_2 is $2 . Og_1$, where g_1 is the middle point of $A_1 A_2$; and since the resultant of $2\, Og_1$ and OA_3 is $3\, Og_2$, where g_2 is determined exactly as in Art. 23, it follows that Leibnitz's graphic representation of the resultant is applicable to non-coplanar forces.

This result follows also analytically; for if (x_1, y_1, z_1), (x_2, y_2, z_2), ... (x_n, y_n, z_n) are the co-ordinates of the extremities

$A_1, A_2, \dots A_n$ of the forces acting on the particle, it is clear that

$$\Sigma X = x_1 + x_2 + \dots + x_n = \Sigma x = n \,.\, \bar{x},$$
$$\Sigma Y = y_1 + y_2 + \dots + y_n = \Sigma y = n \,.\, \bar{y},$$
$$\Sigma Z = z_1 + z_2 + \dots + z_n = \Sigma z = n \,.\, \bar{z};$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ are the co-ordinates of G, the centre of mass of equal masses placed at the extremities of the forces. Hence by equations (1) of Art. 198,

$$R = n \,.\, OG,$$

and $$\cos\theta = \frac{\bar{x}}{OG}, \quad \cos\phi = \frac{\bar{y}}{OG}, \quad \cos\psi = \frac{\bar{z}}{OG},$$

which show that the resultant is represented in magnitude and direction by $n \,.\, OG$.

200.] **Transformation of Couples.** To what has been given in Chapter V on the transformation of couples it is necessary to add a few propositions relating to couples in different planes.

(α) A couple acting on a rigid body may be transferred to any plane parallel to its own.

Let AB (Fig. 229) be the arm of a couple (P, P) and let $A'B'$ be *any* line parallel and equal to AB. At A' introduce two equal and opposite forces, P and P', parallel to AP, and at B introduce the same forces. The introduction of these forces will not disturb the state of the body. Draw AB' and $A'B$, which will bisect each other at O. Then the force P at A and the force P' at B' will give a resultant equal to $2P$ at O; and P at B and P' at A' will give a resultant equal and opposite to this at the same point. Hence there remain the forces P at A' and P at B'; that is, the couple (P, P) with arm AB has been moved to any plane parallel to its own.

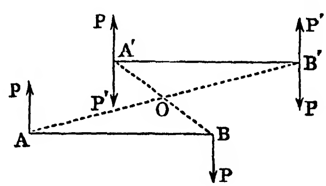

Fig. 229.

From Chapter V it is now clear that the only essential properties of a couple are (1) the constancy of its moment and (2) the direction of its plane; or, in other words, *the constancy of the magnitude and direction of its axis*; the actual *position* of the axis in space is of no consequence, but only its *direction*; two couples whose axes are of equal length and in the same direction are absolutely identical.

Hence the axis of a couple is what is called a *vector*, or directed line of constant magnitude—but not localized—and we shall always, as in the representation of forces, suppose the axis to be marked by an arrow-head.

(β) *Convention with regard to the sense of the axis of a couple.* The following convention for representing the magnitude and sense of the moment of a couple by means of an axis is adopted by common consent for the purpose of enabling us to compound and resolve couples in any planes:—Hold a watch with its plane parallel to the plane of the couple. Then, according as the motion of the hands is contrary to, or along with, the sense in which the couple tends to produce rotation, draw the axis of the couple through the *face* or through the *back* of the watch.

(γ) Two couples result in a single couple whose axis is found from the axes of the component couples by the parallelogram law.

Let the planes of the couples intersect in the line AB (Fig. 230) and the arm of each be made AB, by moving each couple in its own plane, and then suitably altering the forces of each couple (Art. 79, Chap. V). Let P, P be the forces of one couple, and Q, Q those of the other. At B draw* Bp perpendicular to the plane $PABP$ and proportional to the moment of the couple (P, P). We may evidently take $Bp = P$, since the couples have a common arm. Draw Bq perpendicular to the plane $QABQ$ and equal to Q.

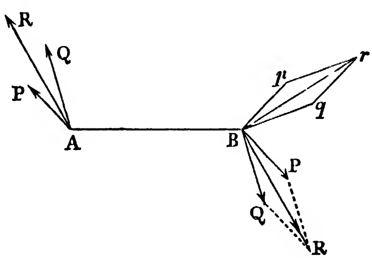

Fig. 230.

Now evidently the forces P and Q at B compound a resultant, R, equal and parallel to the resultant of P and Q at A. Hence the two couples compound a single couple.

Again, draw Br perpendicular to the plane $RABR$ and equal to R. Bp, Bq, and Br are then the axes of the couples (P, P), (Q, Q), and (R, R). But it is manifest that the figure $Bprq$ is

* According to the convention (β) the couples in this figure are both negative, and the axes Bp and Bq should be drawn downwards. This inaccuracy in the figure was detected too late for correction.

merely the figure $BPRQ$ turned round in its own plane through a right angle. Hence Br is the diagonal of the parallelogram determined by the axes of the component couples.

Conversely, any couple may be resolved into two couples whose axes are determined from the axis of the given couple by the parallelogram law; and, as in the case of forces acting at a point, any couple may be resolved into three couples whose axes are determined from the axis of the given couple by the parallelepiped law. All this follows as in Art. 198.

It is well to remark that the axis of a couple represents the moment of the forces of the couple round any line in space parallel to the axis.

(δ) To find the resultant of any number of couples acting in any planes on a rigid body.

Let the axes of the couples be all drawn, each in its proper sense according to the rule (β), at the same point, O (Fig. 228), and let each axis be resolved into three components along rectangular axes Ox, Oy, Oz, drawn through O. Let $L =$ the sum of the axes in the direction Ox; then L is the axis of the component of the resultant couple in the plane yz. Let M and N be the sums of the axes in the directions Oy and Oz, respectively. Then, if G is the resultant axis,

$$G = \sqrt{L^2+M^2+N^2}, \tag{1}$$

and if λ, μ, ν are the direction angles of G,

$$\cos\lambda = \frac{L}{G}, \quad \cos\mu = \frac{M}{G}, \quad \cos\nu = \frac{N}{G}, \tag{2}$$

equations which are exactly analogous to (2) and (3) of Art. 198.

The axes of couples are, therefore, compounded and resolved in the same manner as forces. There is this difference between forces and couples, that, while any number of couples in any planes whatever always result in a single couple, any number of forces cannot, in general, be replaced by a single force, and this difference results from the *vectorial* nature of the axis of a couple.

(ϵ) A force and a couple acting on a rigid body cannot produce equilibrium.

For, let the couple be so transferred that one of its forces, P, acts at a point on the line of action of the force R. Then R

and P at this point compound a single force which, in general, does not intersect the other force of the couple. Therefore, &c.

A force and a couple acting in the same plane are, of course, equivalent to a single force.

201.] **Virtual Work of a Couple.** Let a couple act on a rigid body which receives, or is imagined to receive, any small displacement whatever. It is required to find the work done by the couple in the displacement.

It will be shown subsequently that any displacement of the body may be produced by a motion of translation which is the same for all its points, accompanied by a motion of rotation round an axis through an angle which is the same for all its points.

Now since the forces of the couple are equal and in opposite senses, it is obvious that the sum of their works in any motion of translation is zero.

Again, resolve the motion of rotation into one round an axis perpendicular to the plane of the given couple, and one round an axis in the plane of the couple. It is obvious that the latter displacement will not be productive of work done by the couple, since the forces constituting it may be supposed to act at the points in which they intersect the axis of this component rotation.

There remains only the rotation round an axis perpendicular to the plane of the couple. Suppose O (Fig. 88, Art. 79) to be the point in which this axis intersects the plane of the couple, and let $\delta\theta$ be the angular rotation round the axis, *measured in the sense of the rotation of the couple.* Then the displacements of m and n are $Om \times \delta\theta$ and $On \times \delta\theta$, respectively, so that the work done by the forces is $P\,(Om\,.\,\delta\theta + On\,.\,\delta\theta)$, i.e.

$$P\,.\,h\,.\,\delta\theta, \text{ or } G\,.\,\delta\theta,$$

where $G\,(=P\,.\,h)$ = the moment of the couple.

202.] **Theorem.** *A force acting on a rigid body in a given right line can always be replaced by an equal force acting at any chosen point together with a couple.*

Let a force P (Fig. 231) act at a point A, and let O be the chosen point. At O introduce two forces, P and P', opposite to each other and each equal and parallel to P. Then P at A and P' at O may be taken to constitute a couple whose

moment is Pp, p being the perpendicular from O on the line of action of P at A. There remains, then, the force P at O; and this force together with the couple may replace P at A.

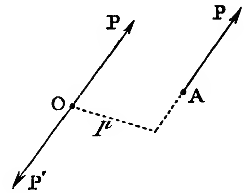

Fig. 231.

Let the axis of this couple be drawn at O; let x, y, z be the co-ordinates of A with respect to a rectangular system of axes drawn through O; and let α, β, γ be the angles which the direction of P makes with the axes of x, y, z, respectively.

The direction-cosines of OA are $\frac{x}{r}$, $\frac{y}{r}$, $\frac{z}{r}$, where $OA = r$, and it is easy to prove that the direction-cosines of the axis of the couple (which is at once at right angles to OA and to P) are

$$\frac{y\cos\gamma - z\cos\beta}{p}, \quad \frac{z\cos\alpha - x\cos\gamma}{p}, \quad \frac{x\cos\beta - y\cos\alpha}{p}.$$

Hence, the axis of the couple being equal to Pp, the projections of the axis on the axes of x, y, and z are

$$P(y\cos\gamma - z\cos\beta), \quad P(z\cos\alpha - x\cos\gamma), \quad P(x\cos\beta - y\cos\alpha);$$

but it is clear from (γ), Art. 200, that these are the axes of the component couples in the planes yz, zx, and xy, into which the couple Pp can be resolved. Putting $P\cos\alpha = X$, $P\cos\beta = Y$, $P\cos\gamma = Z$, we see that the three couples are

$$Zy - Yz, \quad Xz - Zx, \quad Yx - Xy. \tag{1}$$

The force P at O may also be replaced by its three components,

$$X, Y, Z. \tag{2}$$

There is another way in which the reduction is sometimes effected.

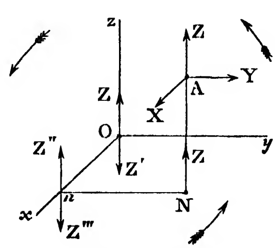

Fig. 232.

Let P at A be resolved into its three components, X, Y, Z, let the line of Z meet the plane (xy) in N, and let Z at A be transferred to N. Let fall Nn perpendicular to Ox; at n introduce two opposite forces Z'' and Z''', each equal and parallel to Z; and at O introduce two opposite forces, Z and Z', each equal and parallel to Z. Now the senses of positive rotation in the planes xy, yz, zx being those indicated by the arrows, the forces Z at N and Z''' at n form a couple whose moment is

$$Zy \text{ parallel to the plane } yz;$$

and the forces Z' at O and Z'' at n form a couple whose moment is

$$-Zx \text{ parallel to the plane } zx;$$

and in addition to these there is the force Z at O.

Similarly, the force X at A can be replaced by X at O together with two couples, Xz and $-Xy$, parallel to the planes zx and xy, respectively; and the force Y at A can be replaced by Y at O together with the couples Yx and $-Yz$ parallel to the planes xy and yz.

Hence P at A is replaced by the forces X, Y, Z at O and the couples $Zy-Yz$, $Xz-Zx$, and $Yx-Xy$, parallel to the planes yz, zx, and xy, respectively.

203.] **Parallel Forces.** Suppose a rigid body to be acted on by any number of parallel forces applied at given points in the body. Take any origin, O, of co-ordinates, and through it draw three rectangular axes, that of z being parallel to the common direction of the forces. Then the force P_1, acting at (x_1, y_1, z_1) may be replaced, as in last Art., by

$$P_1 \text{ at } O \text{ along } Oz,$$

and the couples $\qquad P_1 y_1$ and $-P_1 x_1$

parallel to the planes yz and zx.

Replace each force in this manner: then the whole system will be equivalent to a force

$$P_1+P_2+\ldots+P_n, \text{ or } \Sigma P \text{ at } O,$$

together with the couple

$$P_1 y_1+P_2 y_2+\ldots+P_n y_n, \text{ or } \Sigma Py,$$

parallel to the plane yz, and the couple

$$-P_1 x_1-P_2 x_2-\ldots-P_n x_n, \text{ or } -\Sigma Px,$$

parallel to the plane zx.

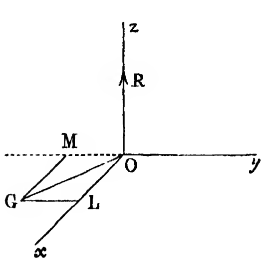

Fig. 233.

These two couples compound a single couple whose axis is found by drawing $OL = \Sigma Py$, on any scale, and OM (in the negative sense of the axis of y) $= \Sigma Px$, on the same scale, and completing the parallelogram $OLGM$ (Fig. 233). If OG, the diagonal, is denoted by G,

$$G = \sqrt{(\Sigma Px)^2+(\Sigma Py)^2}$$

and $\qquad R = \Sigma P,$

R being the resultant force.

204.] **Centre of Parallel Forces.** Since the resultant of two parallel forces, P_1 and P_2, acting at the points A_1 and A_2 divides

the line A_1A_2 in a point g such that $\frac{A_1g}{A_2g} = \frac{P_2}{P_1}$, and since, by elementary geometry (see Art. 84), the distance of g from any plane $= \frac{P_1x_1+P_2x_2}{P_1+P_2}$, where x_1 and x_2 are the distances of A_1 and A_2 from this plane, it follows, by repeating this construction, that the distances, $\bar{x}$, $\bar{y}$, $\bar{z}$, of the *centre of parallel forces* from the planes yz, zx, and xy are given by the equations

$$\bar{x} = \frac{\Sigma Px}{\Sigma P}, \quad \bar{y} = \frac{\Sigma Py}{\Sigma P}, \quad \bar{z} = \frac{\Sigma Pz}{\Sigma P}.$$

205.] **Conditions of Equilibrium of a System of Parallel Forces.** A system of parallel forces has been reduced (Art. 203) to a single force, R, and a single couple, G. Now since these cannot in combination produce equilibrium (ϵ, Art. 200), we must have $R = 0$, and $G = 0$, separately.

Since G cannot be $= 0$ unless $\Sigma Px = 0$ and $\Sigma Py = 0$, the conditions of equilibrium are

$$R = 0, \tag{1}$$

$$\Sigma Px = 0, \ \Sigma Py = 0. \tag{2}$$

DEF. The moment of a force with respect to a plane to which it is parallel is the product of the force and its perpendicular distance from the plane.

Hence for the equilibrium of parallel forces—*The sum of the forces must vanish, and the sum of their moments with respect to every plane parallel to them must also vanish.*

EXAMPLES.

1. A heavy triangular table, ABC, is supported horizontally on three vertical props at the vertices; prove that the pressures on the props are equal.

Let P, Q, R be the pressures at A, B, C, and let a vertical plane through A and the centre of gravity of the table cut the side BC in a, its middle point. For equilibrium the sum of the moments of the forces P, Q, R, and W (the weight of the table) with respect to this plane must $= 0$. But the moments of P and W are each $= 0$, since these forces lie in the plane. Hence the moments of Q and R are equal and opposite. Now $Ba \cdot \sin AaB$ is the distance of Q from the plane, and the distance of $B = Ca \cdot \sin AaC$; and since $Ba = Ca$, these distances are equal. Therefore $Q = R$; and similarly it can be shown that $R = P$; therefore, &c.

2. A heavy triangular plate hangs in a horizontal plane by means of three vertical strings attached to its vertices; at what point in its area must a given weight be placed so that the system of strings may be least likely to break?

Ans. At the centre of gravity of the board. For if $W =$ the weight of the board and P the sustained weight, we have

$$P+Q+R = W+P,$$

or the sum of the tensions is constant, wherever P is placed. Hence if any one is less than $\frac{1}{3}$ $(W+P)$, some other must be greater than this value. It is evident, therefore, that the best arrangement makes each tension $= \frac{1}{3}(W+P)$; but this happens (as proved in last example) when P is placed at the centre of gravity.

3. A heavy elliptic cylinder is sustained in a vertical position by three props applied at three points on the circumference of its base; how should the props be placed in order that the cylinder may be least likely to be upset?

Let the base of the cylinder have any form, ABC (Fig. 234), and let G be the projection of its centre of gravity on the plane of the base. Then, if the props are applied at A, B, and C, the perpendiculars from G on the sides of the triangle ABC must be all equal when the equilibrium is most stable. For, suppose that the cylinder is about to be upset round the line AB; then the moment of the force required to upset it is $W.p$, where W is the weight of the cylinder and p the perpendicular from G on AB. Again, suppose that the cylinder is about to be upset about AC; then the moment of the force required to upset it is $W.q$, where q is the perpendicular from G on AC. Hence if p and q are unequal, advantage will be gained by increasing the smaller of them, even though the greater must be consequently diminished; and it follows that the maximum advantage is gained when p and q are equal. In the same way it can be shown that the perpendicular from G on BC must, in the most advantageous case, be equal to that from G on AB; and therefore the perpendiculars from G on the sides ABC must be all equal.

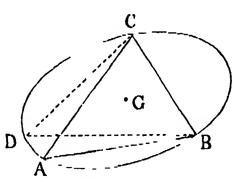

Fig. 234.

Hence the problem amounts to inscribing in a given curve a triangle on the sides of which the perpendiculars from a given point shall be equal. In the particular case in which the base is an ellipse, we have to find a circle concentric with the ellipse, such that a triangle circumscribed to the circle shall be inscribed in the ellipse. Now (Salmon's *Conic Sections*, p. 330, 5th edition), let the ellipse have for equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 = 0$, and the circle $x^2+y^2-r^2 = 0$; then the discriminant of $k(x^2+y^2-r^2)+\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 = 0$ is

$$k^3.r^2+(1+r^2\frac{a^2+b^2}{a^2b^2})k^2+\frac{r^2+a^2+b^2}{a^2b^2}.k+\frac{1}{a^2b^2};$$

and the required condition being $\Theta^2 = 4\,\Delta\,.\,\Theta'$, we have two values for r, namely, $r_1 = \frac{ab}{a+b}$; and $r_2 = \frac{ab}{a-b}$. The first value alone is admissible, because $\frac{ab}{a-b} > b$, and the circle in this case either cuts the ellipse or entirely encloses it.

Since an infinite number of triangles can be inscribed in the ellipse and circumscribed to the circle of radius $\frac{ab}{a+b}$ (Salmon, *ibid.*), the problem is capable of an infinite number of solutions. It is easy to see that in the cases in which it is possible to have a real system of in- and circum-scribed triangles for the ellipse and the circle of radius $\frac{ab}{a-b}$, the centre of the ellipse is outside the area of the triangle. This circle is, therefore, irrelevant to our question.

4. A heavy square board, $ABCD$, of uniform thickness, is hung by three vertical strings, one of which is attached to a corner, A, of the board. The plane of the board being horizontal, find the points, E and F, in the area to which the other two strings should be attached in order that it may be most difficult to overturn the board by placing a weight anywhere on it.

It is evident that advantage is gained by taking the points E and F on the edges of the board.

Assume AE to be the direction of the line joining the points of application of two of the strings, and suppose that a weight, P, placed somewhere in the area ABE is on the point of overturning the board about the line AE. Then the tension of the string at $F = 0$; and if W is the weight of the board, acting at G, the weight P required to upset it is

$$W \times \frac{\text{distance of } G \text{ from } AE}{\text{distance of } P \text{ from } AE}.$$

Hence the greater the distance of P from AE, the less the requisite value of P, or, in other words, the more easily will the board be upset. It is evident, therefore, that the applied weight should be placed at B; and in the same way, if the board is to be upset round the lines AF and FE, the applied weights should be placed at the corners D and C, respectively.

Again, in the arrangement of greatest advantage, the board is upset with equal ease round each of the lines AE, AF, and FE. For, if it be more easily upset round one of these lines than round another, advantage will be gained by making it a little more stable with regard to the first. Hence, since the weights placed at B, D, and C are all equal, we have

$$\frac{\text{distance of } G \text{ from } AE}{\text{distance of } B \text{ from } AE} = \frac{\text{distance of } G \text{ from } AF}{\text{distance of } D \text{ from } AF} = \frac{\text{distance of } G \text{ from } EF}{\text{distance of } C \text{ from } EF}.$$

The angles EAB and FAD are, therefore, equal, and each

$$= \tan^{-1}(\sqrt{2}-1).$$

5. A heavy elliptic table is supported on three vertical props; how must they be placed so that it may be most difficult to upset the table by placing a weight on it?

Ans. The props must be placed at three points, A, B, C, on the circumference of the ellipse; and if γ is the eccentric angle of C, that of B is $\frac{2}{3}\pi+\gamma$, and that of A is $\frac{4}{3}\pi+\gamma$. The weight which, most advantageously applied, will then just upset the table is half its own weight.

This may be seen as follows. Assume any line in the area as the line joining two props: then the least weight that will be required to upset the table must be placed at the point of contact of a tangent parallel to the assumed line, since the weight will have maximum leverage at this point. Also, it must be equally easy to upset the table round the three lines AB, BC, CA; that is, the requisite weights placed at C', A', B', the points of contact of the tangents, must be all equal. If, then, x, y, z are the perpendiculars from the centre on the lines BC, CA, AB, and P, Q, R the perpendiculars on the parallel tangents, we must have

$$\frac{x}{P-x} = \frac{y}{Q-y} = \frac{z}{R-z};$$

or if α, β, γ are the eccentric angles of A, B, C,

$$\frac{\cos\frac{1}{2}(\alpha-\beta)}{1-\cos\frac{1}{2}(\alpha-\beta)} = \frac{\cos\frac{1}{2}(\beta-\gamma)}{1-\cos\frac{1}{2}(\beta-\gamma)} = \frac{-\cos\frac{1}{2}(\alpha-\gamma)}{1+\cos\frac{1}{2}(\alpha-\gamma)},$$

a negative sign being used in the last, since (γ, β, α being in ascending order of magnitude) $\frac{1}{2}(\alpha-\gamma)$ is evidently $>\frac{1}{2}\pi$. Hence $\beta = \frac{2}{3}\pi+\gamma$, $\alpha = \frac{4}{3}\pi+\gamma$; and the weight required to upset the table $= W\frac{x}{P-x}$, or $\frac{1}{2}.W$. Any one position of C is, therefore, as good as any other; and if C is made the extremity of either axis, the line AB is parallel to the other at a distance equal to $\frac{1}{4}$ of the first axis from it.

6. A rectangular board is held with its plane horizontal by three vertical strings attached to three of its corners; find the point in its area at which a weight must be placed so that the tensions of the strings shall be given multiples of the weight of the board.

Ans. Let W be the weight of the board; let the strings be applied at the corners A, B, C; let $AC = 2a$, $AB = 2b$; and let the tensions of the strings at A, B, C be lW, mW, nW, respectively.

Then the weight must be placed at a point whose distances from AB and AC are

$$\frac{2n-1}{l+m+n-1} \cdot a \quad \text{and} \quad \frac{2m-1}{l+m+n-1} \cdot b.$$

The magnitude of the weight is, of course, $(l+m+n-1)\ W$.

7. A uniform circular lamina is placed with its centre upon a prop; find at what points on its circumference three weights, w_1, w_2, w_3, must be placed that it may remain at rest in a horizontal position (Walton's *Mechanical Problems*, p. 73).

Ans. The angles which the weights subtend in pairs at the centre of the lamina are the supplements of the angles of a triangle whose sides are proportional to the weights.

206.] Reduction of a System of Forces acting in any manner on a Rigid Body. Let any origin, O (Fig. 232), be assumed arbitrarily, and let any system of rectangular axes, Ox, Oy, and Oz, be drawn through it. If, then, forces $P_1, P_2, P_3, \ldots$ act on the body at points whose co-ordinates are (x_1, y_1, z_1), (x_2, y_2, z_2), $(x_3, y_3, z_3), \ldots$ each force can be replaced by three components acting at O along the axes, together with three couples whose axes coincide with the co-ordinate axes. The force P_1, for example, is equivalent to X_1, Y_1, Z_1 at O and three couples, $Z_1 y_1 - Y_1 z_1$, $X_1 z_1 - Z_1 x_1$, and $Y_1 x_1 - X_1 y_1$. Add the components of the forces, and also the axes of the couples, in the directions Ox, Oy, and Oz: the whole system of forces is equivalent to

the force ΣX along Ox,

„ ΣY „ Oy,

and „ ΣZ „ Oz;

and the system of couples is equivalent to

the couple $\Sigma\,(Zy - Yz)$, or L, in the plane yz,

„ $\Sigma\,(Xz - Zx)$, or M, „ zx,

and „ $\Sigma\,(Yx - Xy)$, or N, „ xy.

(Of course the axes of L, M, N are drawn along the axes of x, y, and z, respectively.)

Hence if R is the magnitude of the resultant of translation,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2};$$

and if G is the magnitude of the resultant couple,

$$G = \sqrt{L^2 + M^2 + N^2}.$$

The direction-cosines of R are $\frac{\Sigma X}{R}$, $\frac{\Sigma Y}{R}$, and $\frac{\Sigma Z}{R}$; and those of G are $\frac{L}{G}$, $\frac{M}{G}$, and $\frac{N}{G}$.

Thus, *any system of forces acting on a rigid body can be replaced by a single resultant force acting at an arbitrary origin, the magnitude and direction of this force being the same for all origins, and a single resultant couple the magnitude and direction of whose axis are both dependent on the origin chosen.*

It has been already remarked (Art. 200) that G is not only the axis of the resultant couple (corresponding to a resultant force acting at O), but also the sum of the moments of the forces about a line at O drawn in the direction of G; and since the axes of couples have been proved to follow the parallelepiped and parallelogram laws, it follows that the sum of the moments of the forces about this line is greater than the sum of their moments about any other line at O; and also that the sum of the moments of the forces about any other line through O is the resolved part of G in the direction of this line.

Remark. The magnitude and direction of G are constant at all points along the same right line parallel to R. For R may be supposed to act at any point in this line, and the vector G may be moved parallel to itself to the point at which R is supposed to act. The axis G is called the *axis of principal moment* at O.

207.] **Moment of a Force round any Line.** Let a force of given magnitude act in a given direction at a given point A, and let its moment be required about a given right line passing through a given point P. With reference to any three rectangular axes, let (x, y, z) be the co-ordinates of A; let (ξ, η, ζ) be those of P; and let (X, Y, Z) be the components of the force.

Then the moment of the force round a line through A parallel to the axis of x is

$$Z(y-\eta)-Y(z-\zeta),$$

while its moments round the lines through A parallel to the axes of y and z are, respectively,

$$X(z-\zeta)-Z(x-\xi) \text{ and } Y(x-\xi)-X(y-\eta).$$

Denote these component moments by L, M, N, respectively. Then, if the line through A about which the total moment is

required makes angles whose direction-cosines with the axes of reference are l, m, n, the required moment is

$$lL + mM + nN.$$

208.] **Poinsot's Central Axis.** Any system of forces acting on a rigid body has been proved to be equivalent to a single resultant force, R, acting at an arbitrary origin, O, and a single resultant couple, G. Let ϕ be the angle between R and G, and resolve G into two components, OK and On (Fig. 235) along and perpendicular to R, respectively. On is the axis of a couple in the plane ROx, perpendicular to On. Now let each force of this couple be made equal to R, and the arm, OP *, is consequently equal to $\frac{On}{R}$; that is,

$$OP = \frac{G \sin \phi}{R}. \tag{1}$$

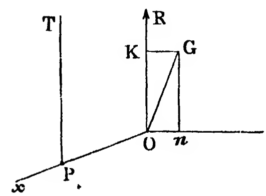

Fig. 235.

One of these forces may be applied at O to destroy the resultant, R, at this point, and there finally remains a resultant force, R, at P along PT (parallel to OR), together with a couple whose axis is OK, or $G \cos \phi$. Denoting OK by K, we have then

$$K = G \cos \phi. \tag{2}$$

The axis OK may, of course, be drawn at P along PT [(α), Art. 200].

Hence *the whole system of forces is equivalent to a resultant force equal to R acting along the line PT and a couple in a plane perpendicular to this line.*

The line PT thus determined is called *Poinsot's Central Axis.*

To construct Poinsot's Central Axis for any system of forces—*Reduce the forces to a resultant force, OR, acting at any origin, O, and a couple whose axis is OG; then on a line perpendicular to the plane of OR and OG measure off a length, OP* †, *equal to* $\frac{G \sin \phi}{R}$, *where* ϕ *is the angle between OR and OG. A line through the point P parallel to OR is the required Central Axis.*

* The point P should be represented on the production of the line xO through O, according to the convention of Art. 200. The inaccuracy in the figure was detected too late for correction.

† The *sense* of OP is determined by the convention of Art. 200.

209.] **Theorem.** A given system of forces has but one Central Axis.

This important proposition may be proved *ex absurdo* thus:—

Whatever origin be chosen, the resultant force acting at it is constant both in magnitude and in direction. Now, if it be possible, let the system of forces be equivalent to a resultant R acting at O and a couple whose axis is OK, and also to a resultant force R acting at O' and a couple whose axis is $O'K'$, the lines OK and $O'K'$ being, of course, in the direction of R. Then it is evident that the force R at O and the couple OK should equilibrate the *reversed* force R and *reversed* couple $O'K'$, at O'. But the couples give a single couple, $OK \backsim O'K'$, and the forces give also a couple which, being in a plane perpendicular to the first couple, cannot with it produce equilibrium. Therefore, &c.

Since this axis is unique, equation (2) of the last Article shows that for all origins the quantity $G \cos \phi$, or the *projection of the axis of the resultant couple along the direction of the resultant force is constant, and equal to Poinsot's moment.*

210.] **Theorem.** The sum of the moments of the forces round Poinsot's Axis is less than the sum of their moments round any other axis of principal moment. This is, of course, evident from what has just been said. The deduction of the axis of principal moment at any point from Poinsot's may, however, for clearness, be specially exhibited.

Let Oz (Fig. 236) be Poinsot's Axis and OK $(= K)$ the moment of the forces round it. Let O' be any point in the body, and let it be proposed to find the principal moment at this point; $O'O$ is a line drawn through O' perpendicular to Poinsot's Axis. At O' introduce two equal and opposite forces, $O'R$ and $O'R'$, each $= R$. Then OR and $O'R'$ form a couple, whose axis, $O'n$, is perpendicular to the plane $ROO'R'$ and equal to $R \,.\, OO'$. Transfer the axis OK to $O'K'$ (Art. 200), and draw the diagonal, $O'G$, of the rectangle determined by $O'n$ and $O'K'$. Then $O'G$ $(= G)$ is the axis of principal moment at O', and it is evidently $> O'K'$. Hence Poinsot's is the least principal moment.

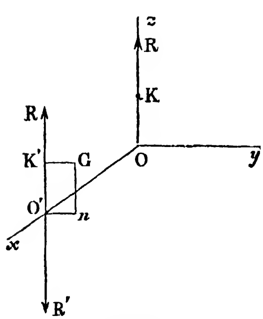

Fig. 236.

211.] **Definition of a Wrench.** It has just been shown that any given system of forces acting on a rigid body can be reduced to a single force, R, and a single couple, K, such that the axis of the couple is coincident with the line of action of the force, and that this reduction, for the given force system, is unique.

A force acting along a line and a couple whose axis coincides with this line constitute together what is called a *wrench.*

The ratio of the moment of the couple, K, to the magnitude of the force, R, is evidently a *linear magnitude*, and is called *pitch.*

The right line about which the wrench takes place, when contemplated in conjunction with the pitch, is called a *screw.*

Thus, then, *a screw is a definite right line in space associated with a definite pitch.*

The right line itself about which the wrench takes place—the axis of the wrench—may be denoted by the symbol α, and the pitch associated with it may be denoted by the symbol p_α. It is evident that the complete determination of a screw (pitch included) requires *five* constants, since the axis may be determined by two equations of the forms

$$x = az + m, \; y = bz + n,$$

which involve the four constants a, m, b, n; while the pitch is specified by another constant.

When the force and the axis of the couple—this latter drawn according to the convention of Art. 200—are in the same sense along the axis of the wrench, the pitch is positive; when they are in opposite senses, it is negative.

The force which acts in a wrench is called by Sir R. Ball the *intensity* of the wrench.

A force alone may be regarded as a wrench of zero pitch.

A couple alone may be regarded as a wrench of infinite pitch.

212.] **Wrench of Two Forces.** Let it be required to find the wrench of which two forces, P and Q, represented in magnitudes and lines of action by the two non-intersecting lines AP and BQ (Fig. 237), are equivalent.

Let AB be the shortest distance between the lines of action of the two given forces, and denote the length AB by h.

Then, following the rule of Art. 208, reduce the forces to a resultant acting at A together with a couple, by introducing two forces, Aq and Aq', equal, opposite, and parallel to Q. Com-

pounding AP and Aq, we get $Ar = R$ = resultant force; and drawing, in the sense determined by the convention of Art. 200, An to represent $Q \times h$, the moment of the couple (BQ, Aq'), the force and couple for the origin A are Ar and An. If O (on AB) is any point on AB, the couple corresponding to it is got by combining with An the axis Am which represents $R \times AO$, this line Am being perpendicular to Ar and in the plane of the lines An, AP, Aq; and if O is Poinsot's origin, the resultant, Ac, of An and Am coincides in direction with Ar.

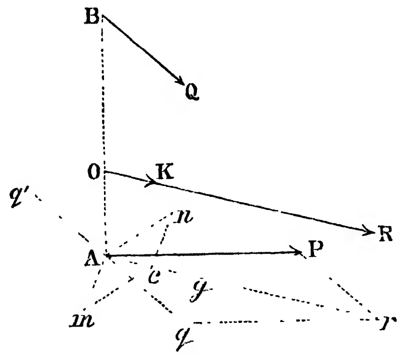

Fig. 237.

Now since Acn is a right angle,

$$\cos Anc = \frac{cn}{An} = \frac{R \times AO}{Q \times h}.$$

But $\quad \angle Anc = \angle qAr; \qquad \therefore \quad Q \times \cos Anc = Ag,$

where g is the foot of the perpendicular from q on Ar. Hence

$$\frac{AO}{h} = \frac{Ag}{Ar},$$

or $$\frac{AO}{OB} = \frac{Ag}{gr} = \frac{\text{projection of } Q \text{ along } R}{\text{projection of } P \text{ along } R};$$

so that *Poinsot's origin divides the shortest distance between P and Q inversely as the orthogonal projections of these forces along the direction of their resultant of translation.* Or, again, O may be determined by drawing from g the line gO parallel to the line rB.

The wrench to which P and Q are equivalent is represented in the figure by (OR, OK).

213.] **Two Intersecting Rectangular Screws.** Suppose OX (Fig. 238) to be the axis of a wrench the force in which is represented by the length OX $(= X)$, and the moment of the couple by OM. If p_x is the pitch of this screw, the moment of the couple is

$$p_x \cdot X.$$

Also, let OY and OL represent the force and couple in another wrench intersecting OX at right angles, and let p_y be the pitch of this second screw.

It is required to find the resultant wrench to which these two wrenches are equivalent.

Replace the forces X and Y by their resultant, OD; and also resolve the moments, OM and OL, into components along and perpendicular to OD.

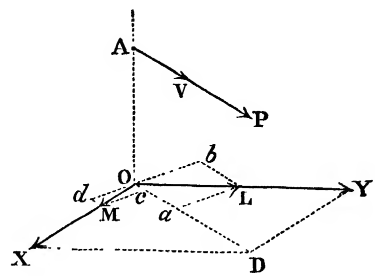

Fig. 238.

If θ denotes the angle DOX, we shall have along OD a moment, $Oa + Oc$, equal to $p_x \,.\, X \cos\theta + p_y \,.\, Y \sin\theta$; or if $OD = P$, we have along OD

$P\,(p_x \,.\, \cos^2\theta + p_y \,.\, \sin^2\theta)$.

Perpendicular to OD, the resultant moment is $Ob - Od$, or

$$P\,(p_y - p_x) \sin\theta \cos\theta.$$

Now a force OD $(= P)$ and a couple whose axis, $Ob - Od$, is perpendicular to it are equivalent to a force equal and parallel to OD at a distance, OA, from OD, along the perpendicular to the plane of OD and the axis of the couple, such that

$$P \times OA = Ob - Od = P\,(p_y - p_x) \sin\theta \cos\theta\,;$$

$$\therefore \quad OA = (p_y - p_x) \sin\theta \cos\theta. \qquad (1)$$

Hence the two given wrenches are equivalent to the wrench consisting of the force P at A and the couple whose axis $AV = P\,(p_x \,.\, \cos^2\theta + p_y \,.\, \sin^2\theta)$; so that if p_θ denotes the pitch of the resultant screw,

$$p_\theta = p_x . \cos^2\theta + p_y . \sin^2\theta. \qquad (2)$$

The whole process may, of course, be reversed; i.e. starting with the single wrench about the screw AP, we may resolve it in an infinite number of ways into a pair of wrenches about two intersecting rectangular screws. The positions of these screws may be assigned by the distance OA and the angle θ; and when this is done, the component pitches, p_x and p_y, are given by (1) and (2).

214.] **The Cylindroid.** *Given two intersecting rectangular screws, it is required to find the locus of all screws which result from wrenches of any variable intensities about these two given screws.* That is, given two right lines, OX and OY, and two linear constants, p_x and p_y, associated with them, if a wrench in which the force is X and the couple $p_x \,.\, X$ act about OX, the magnitude X being anything whatever; and if a wrench in

which the force is Y and the couple $p_y . Y$ act about OY, the magnitude Y being also anything whatever; find what surface is traced out by the axis of the resultant wrench, as X and Y are separately or simultaneously varied in any manner.

Taking OX and OY (Fig. 238) as axes of x and y, and OA, their common perpendicular, as axis of z, the equations of AP are obviously

$$z = (p_y - p_x)\sin\theta\cos\theta, \tag{1}$$

$$y = x\tan\theta, \tag{2}$$

the angle θ depending on the magnitudes X and Y.

Hence, whatever θ may be, we have

$$z(x^2+y^2) - (p_y - p_x)xy = 0, \tag{α}$$

which is the equation of the surface traced out by the line AP as X and Y are varied. This surface is called the *Cylindroid*.

215.] **To construct the Cylindroid.** Easy methods of constructing the cylindroid at once present themselves. It is sufficient to give one. Taking two rectangular axes, Ox and Oy, and a perpendicular, Oz, to them, we are to imagine a right line which begins by lying along Ox to travel up along Oz, while it always remains parallel to the plane xy and rotates round Oz, the angle, θ, through which it has rotated, and the corresponding distance, z, through which it has risen being connected by equation (1) of last Article.

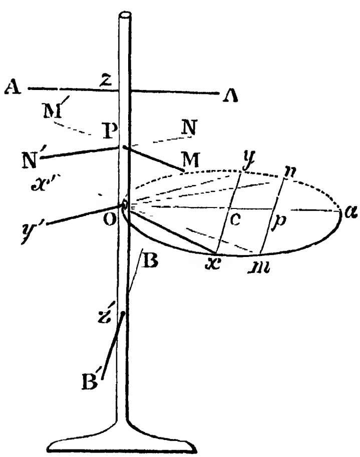

Fig. 239.

Let PM (Fig. 239) be any position of the moving line, its projection on the plane of xy being Om, and $\angle mOx = \theta$. Then, putting $p_y - p_x = 2h$, we have

$$OP = h\sin 2\theta.$$

Draw Oa bisecting the angle xOy, and equal to $2h$, and describe a circle on Oa as diameter, c being its centre, and Om meeting it in m. From m draw the chord mpn perpendicular to Oa. Then $\sin 2\theta = \cos mcp$; $\therefore$ $OP = cp$.

When $\theta = \frac{1}{4}\pi = xOa$, OP is greatest and equal to h, and the moving line, zA, is then parallel to Oa, the distance Oz being equal to h. Hence $Pz = ap$.

Thus we get a simple method of constructing the surface:— Divide up the whole diameter aO into any number of parts, ap, &c. (equal for simplicity). On the axis, Oz, take the length $Oz = ca =$ radius of circle; beginning with the point z, measure off parts, zP, &c., successively equal to the parts ap, &c.; then through any point, P, on Oz draw two parallels, PM and PN, to the lines Om and On, joining O to the extremities of the corresponding chord of the circle.

The ruled surface traced out thus by all the pairs of lines, such as PM and PN, is the cylindroid.

It is obvious, of course, from the equation $z = h \sin 2\theta$, that through each point P on the axis Oz there are two generators, which coincide at the point z with a parallel to Oa. When P moves upwards from O along Oz, θ runs from 0 to $\frac{1}{4}\pi$, until z is reached; when θ increases beyond $\frac{1}{4}\pi$, the moving point P descends from z towards O, and in its descent gives the second generator PN at P, which is parallel to On. When $\theta = \frac{1}{2}\pi$, P is at O and the generator is Oy. As θ increases beyond $\frac{1}{2}\pi$, the moving point P travels downwards, along Oz', until $\theta = \frac{3}{4}\pi$, when z' is reached, Oz' being equal to h, and the generator being $z'B'$, which is parallel to the tangent at O to the circle. As θ increases beyond $\frac{3}{4}\pi$, the moving point moves up again towards O, which it reaches when $\theta = \pi$, the generator then coinciding with Ox, its original position. Thus all through the motion the generator has continuously revolved in the same sense—counter clockwise.

Another way of looking at the matter is this—imagine a pair of scissors placed with the rivet at z and the blades closed and coinciding with $A'zA$; then let the rivet be gradually brought down along zO while the blades gradually open in such a way that when they are parallel to a pair of chords Om and On, the rivet has descended through a distance equal to ap. (A vivid figure of the cylindroid will be found in Ball's *Theory of Screws*.)

216.] **Angle between Two Screws.** In order to make our equations in the sequel universally applicable without ambiguity, it becomes necessary to give a definite meaning to the angle between two screws, since *à priori* the expression is not definite.

The following definition of the angle between two screws will be found to be of universal application whether the pitches are both positive, or both negative, or one positive and the other negative :—

Let the axis of each screw be marked with an arrow-head pointing in the sense in which the *force* acts along the screw. The two screws being denoted by α and β, place a watch with its back towards α and its face towards β, the shortest distance between them passing perpendicularly through its face. Then the angle through which the arrow on α must be rotated, in a sense opposite to that of the watch-hand rotation, so that this arrow shall be parallel to and in the sense of the arrow on β, is the angle between the screws.

217.] **Theorem.** *Any two given screws determine a cylindroid.* Let AP (Fig. 238) and BQ* be the axes of any two given screws whose pitches are, respectively, p_θ and p_ϕ, the line AB being the shortest distance between them. Let $AB = h$. Then what we have to show is that it is possible to find a single pair of rectangular lines, OX and OY, such that if the wrench of pitch p about AP is resolved into two wrenches about these lines, and if the wrench of pitch p_ϕ about BQ is also resolved into two wrenches about OX and OY, we shall get the same value in each case for the pitch about OX and also the same value for the pitch about OY.

Let ω be the angle between AP and BQ; let AP and BQ make angles θ and ϕ with the sought line OX, the point O being on AB at a distance z from B; and assume that each resolution gives a pitch p_x about OX, and a pitch p_y about OY. Then we have

$$p_\theta = p_x \,.\, \cos^2\theta + p_y \,.\, \sin^2\theta, \quad p_\phi = p_x \,.\, \cos^2\phi + p_y \,.\, \sin^2\phi\,; \qquad (1)$$

$$z + h = (p_y - p_x)\sin\theta\cos\theta, \qquad z = (p_y - p_x)\sin\phi\cos\phi, \qquad (2)$$

where $\theta = \omega + \phi$. Hence

$$p_\theta + p_\phi = p_y + p_x - (p_y - p_x)\cos\omega\cos(\omega + 2\phi)\,; \qquad (3)$$

$$p_\theta - p_\phi = (p_y - p_x)\sin\omega\sin(\omega + 2\phi)\,; \qquad (4)$$

$$h = (p_y - p_x)\sin\omega\cos(\omega + 2\phi)\,; \qquad (5)$$

---

* BQ is not represented in the figure; but, for definiteness, B is supposed to lie on OA between O and A, while the projection of BQ on the plane OXY lies between OX and OD.

so that we have

$$\tan(\omega + 2\phi) = \frac{p_\theta - p_\phi}{h}; \tag{6}$$

$$p_y + p_x = p_\theta + p_\phi + h\cot\omega; \tag{7}$$

$$p_y - p_x = \sqrt{h^2 + (p_\theta - p_\phi)^2}\,\mathrm{cosec}\,\omega; \tag{8}$$

which give definite values for p_x, p_y, and ϕ. Hence the lines OX and OY can be determined, and therefore also a single cylindroid containing the two given screws.

218.] **Composition of Wrenches.** *The resultant of any two wrenches is a wrench about a screw on the cylindroid determined by the two given wrenches.*

For, let p_1 and P_1 be the pitch and intensity of one, and p_2 and P_2 the pitch and intensity of the other. Also let p_x and p_y be the pitches of the two principal (or rectangular) screws, Ox and Oy, of the cylindroid. Then (Art. 213) replacing the first wrench by its components round Ox and Oy, we get a moment $p_x . X_1$ round Ox, and a moment $p_y . Y_1$ round Oy, the components of P_1 parallel to Ox and Oy being X_1 and Y_1. Similarly, replacing the second wrench by its components, we have finally the moments

$$p_x(X_1 + X_2) \text{ and } p_y(Y_1 + Y_2)$$

round Ox and Oy, respectively. But if we take the resultant of the forces P_1 and P_2, as if they acted at a point, and if its components parallel to Ox and Oy are X and Y, we know that $X = X_1 + X_2$, and $Y = Y_1 + Y_2$. Therefore round Ox and Oy we have simply wrenches of intensities X and Y, which (Art. 213) give a single wrench about that screw on the cylindroid which is parallel to the direction of the resultant of translation of the given forces P_1 and P_2.

Hence the proposition of the *parallelogram of forces* for forces acting at a point becomes simply a proposition of the *parallelogram of screws* for the composition of wrenches.

Hence also three wrenches will be in equilibrium if they take place about three screws on the same cylindroid, whose directions are so related that the intensity of the wrench on any one screw is proportional to the sine of the angle between the directions of the other two screws—the well-known *law of Sines*.

And, generally, the resultant of any number of wrenches about screws situated on the same cylindroid may be found

by transferring all the forces in the wrenches to a single point, finding the resultant of these forces, and taking the screw on the cylindroid which is parallel to the direction of this resultant. A wrench about this screw with intensity equal to the resultant force is the resultant wrench sought.

COR. A wrench about any given screw on a cylindroid can be resolved into wrenches about any two assigned screws on the same cylindroid. For, a force acting along any given line can be resolved into two components along any two lines which meet it if they all lie in the same plane. In this way the intensities of the two component wrenches along the two assigned screws are determined.

219.] **Distribution of Pitch.** The pitches belonging to the various screws on a cylindroid may be graphically represented thus.

Taking the two principal screws of the cylindroid as axes, construct the conic whose equation is

$$x^2 . p_x + y^2 . p_y = k^3, \tag{1}$$

where k is any constant length. If r is the radius of this conic making an angle θ with the axis of x (i. e. the screw of pitch p_x), we have

$$p_x . \cos^2\theta + p_y . \sin^2\theta = \frac{k^3}{r^2}. \tag{2}$$

But by (2) of Art. 213 the left-hand side is the pitch of the screw whose axis is parallel to r. Hence

$$p_\theta = \frac{k^3}{r^2}, \tag{3}$$

which graphically represents p_θ in precisely the same way as the moment of inertia of a lamina is represented.

The conic (1) is called the *pitch conic* of the cylindroid. It is an ellipse if the principal pitches have the same sign, and a hyperbola if they have opposite signs.

In the latter case there will be two screws of zero pitch, viz., those parallel to the asymptotes of the pitch hyperbola. In every case there will be *two* screws having a given pitch, and they are parallel to two equal diameters of the pitch conic.

This conic possesses the following noteworthy property. *If the wrench on any screw of the cylindroid is replaced by a force and a couple at the centre of the pitch conic (centre of the cylindroid), the axis of this couple will lie along the perpendicular to*

the diameter of the pitch conic which is conjugate to the direction of the force—or, in other words, the plane of the couple will be that of the axis of the cylindroid and this conjugate diameter.

For, let the axis of the given wrench make an angle θ with the axis of x at the centre, O, of the cylindroid, and let P be the intensity of the wrench. Then the component wrenches at O to which the given one is equivalent are $(P\cos\theta,\ Pp_x\cos\theta)$ and $(P\sin\theta,\ Pp_y\sin\theta)$. The two couples, $Pp_x\cos\theta$ and $Pp_y\sin\theta$, at O compound into a couple, G, making with Ox an angle ψ such that $\cot\psi = \frac{p_x}{p_y}\cot\theta$. Hence

$$\tan\theta\tan(\tfrac{1}{2}\pi+\psi) = -\frac{p_x}{p_y},$$

which is the well-known equation connecting the directions of two conjugate diameters of the conic, the squares of whose axes are $\frac{k^3}{p_x}$ and $\frac{k^3}{p_y}$, so that the line perpendicular to G is the diameter conjugate to the direction of the given screw (θ).

220.] **Screw Motion of a Rigid Body.** It will be shown in a subsequent chapter that if a rigid body occupying a position which we may denote by (A), be displaced in any manner so as to occupy another position (B), the change from (A) to (B) could have been effected by rotating the body round a certain axis, and then giving it a motion of translation along this axis; in other words, Poinsot's result for a system of forces holds for the displacements of the individual points of a rigid body—viz., *the displacement can be produced by giving the body a twist* about a screw.*

The ratio of the motion of translation along the axis of the screw to the circular measure of the angle of rotation about it is called the *pitch* of the screw; so that, as in the case of forces and couples, *the pitch is still a linear magnitude.*

A motion of translation alone may be regarded as a twist of infinite pitch.

---

* Sir R. Ball uses this term *twist* to denote a rigid-body motion which consists of a translation along, accompanied by a rotation about, a line. The term *twist* is, however, so generally used to signify a *strain* of a natural solid—which is wholly distinct from a rigid-body motion—that it is advisable to call the attention of the student to its technical signification in Sir R. Ball's theory.

A motion of rotation alone may be regarded as a twist of zero pitch.

227.] **Degrees of Freedom of a Rigid Body.** The position of a rigid body in space is completely defined by *six* independent variables, viz., the three co-ordinates of some point in it with reference to assumed rectangular axes, and the three angles which in the well-known theory of the motion of a rigid body about a fixed point determine the positions of all points in the body relatively to this fixed point. The body may, however, be so hampered in any case that these six variables are not all independent. If each of them may be anything whatever independently of any of the others, the body is perfectly free or has *freedom of the sixth order*, or *six degrees of freedom.* If the variables are connected by one equation, so that virtually only five are independent (the sixth being known as soon as any five are assumed), the body has *freedom of the fifth order*, or *five degrees of freedom.* If they are connected by two equations, or, in other words, if the position of the body depends on only *three* independent variables, the body has *freedom of the third order*, or *three degrees of freedom*; and so on.

A rigid body occupying any position can be brought into an indefinitely near position by giving it a small motion of translation whose components parallel to fixed rectangular axes are $(\delta a, \delta b, \delta c)$, which are the components of the translation of any point, A, in the body, and rotating it round axes of reference at A parallel to the fixed axes through angles $(\delta\theta_1, \delta\theta_2, \delta\theta_3)$.

The component absolute motions of any point in the body are expressed by equations of the form

$$\delta x = \delta a + (z-c)\,\delta\theta_2 - (y-b)\,\delta\theta_3, \text{ \&c.},$$

(a, b, c) being the co-ordinates of A with reference to the axes through the fixed origin.

EXAMPLES.

1. The sum of the pitches of the two screws which pass through any point on the axis of a cylindroid is constant.

2. A cubical block (represented by Fig. 228) is free to twist about its diagonal OO'; determine a wrench—

(*a*) about AB,
(*b*) about AD,

so that the block may be in equilibrium.

Ans. In (*a*) the wrench is one of infinite pitch, i. e. a couple about AB. In (*b*) the pitch of the screw on OO' being p, that of the screw on AD is $a-p$, where a is the length of an edge of the block, so that the wrench is $[P, (a-p)P]$, where P is a force of any magnitude.

3. A right cone is capable of twisting about a screw coincident with one of its generating lines; find the wrench about a given diameter of its base which will keep it in equilibrium.

Ans. If the axis of the given screw of twist (pitch p) is BA, where B is the vertex and A a point on the circumference of the base, O the centre of the base, OP the radius of the base about which the wrench is to take place, P being on the circumference of the base, $\angle POA = \theta$, c = height of cone, the required wrench is

$$[P, -(p+c\tan\theta)P].$$

4. A body which has freedom of the second order is acted upon—

(*a*) by a single force,

(*b*) by a single couple;

what is the condition of equilibrium?

Ans. In case (*a*) the line of action of the force must intersect both the screws of zero pitch on the cylindroid which defines the possible motions of the body; and in (*b*) the axis of the couple must be parallel to the nodal line of the cylindroid.

5. At a given point, O, are compounded three wrenches of fixed pitches, a, b, c, along three fixed rectangular lines, Ox, Oy, Oz; the intensities of these wrenches being all varied in any manner, find the surface-locus traced out by the Poinsot centre.

Ans. Its equation is

$$\frac{\overline{b-c}^2}{x^2} + \frac{\overline{c-a}^2}{y^2} + \frac{\overline{a-b}^2}{z^2} + \frac{\overline{a-b}\,.\,\overline{b-c}\,.\,\overline{c-a}}{xyz} = 0.$$

The section of this surface by any plane through any axis of coordinates is an ellipse (and the axis itself). The force and the principal couple at O are always related thus—the force being a central radius vector of a fixed ellipsoid, the axis of the principal couple coincides in direction with the central perpendicular on the tangent plane to this ellipsoid at the extremity of the radius vector, and varies inversely as this perpendicular.

235.] **Theorem.** A system of forces can be reduced to two forces in an infinite number of ways. For they can be reduced to a resultant force, R, acting at any point, together with a couple. Now the forces of the couple can be made of any magnitude by varying its arm; and one of them can be combined with R. There will then remain the resultant of R and this force together with the remaining force of the couple. Therefore, &c.

Of course the wrench to which all pairs of forces equivalent to a given force system reduce is unique; and since we have shown (Art. 212) that the wrench of two forces takes place about a screw which intersects the shortest distance between the lines of action of the two forces, we see that—*Poinsot's axis intersects the shortest distance between the lines of action of every pair of forces to which the given force system can be reduced.*

Suppose that AP and BQ (Fig. 237) are a pair of forces to which a given force system can be reduced, and let $p = \frac{K}{R}$ = the pitch of the Poinsot screw to which they are equivalent. Then the distance (OA) of either line (AP) from the Poinsot axis is the product of the pitch and the cotangent of the inclination of the other line (BQ) to the Poinsot axis.

For if $\theta = PAr$, $\phi = pAr$, since Ac represents K and An represents Qh, we have $K = q \,.\, R = Qh \sin\phi$. But

$$\frac{AO}{OB} = \frac{Q\cos\phi}{P\cos\theta}; \qquad \therefore\ \frac{AO}{h} = \frac{Q\cos\phi}{R};$$

$$\therefore\ AO \times R = Qh\cos\phi;$$

$$\therefore\ AO = p\cot\phi.$$

Similarly $$BO = p\cot\theta.$$

The two lines of action, AP, BQ, of any pair of forces equivalent to a given wrench are sometimes called *reciprocal lines*.

They possess the following property—if any point, S, be taken on either line (AP), the axis of principal moment at this point is the perpendicular to the plane containing S and the other line (BQ).

This property is at once obvious, since to get G, the axis of principal moment at S (supposed on AP), we introduce at S two forces equal and opposite to Q; then the couple Q at B and $-Q$ at S is in the plane of S and BQ, and its axis, G, is, of course, perpendicular to this plane.

The relation between the two lines is thus reciprocal, so that either line is the envelope of the planes of principal couples at all points on the other line.

The two forces P and Q along AP and AQ may, of course, be regarded as two wrenches each of zero pitch, and therefore as determining a cylindroid. If in Art. 217 we put $p_\theta = p_\phi = 0$, we find $p_y = h\cot\frac{1}{2}\omega$, $p_x = -h\tan\frac{1}{2}\omega$; also the origin of the

cylindroid bisects the distance h, and its axes are parallel to the internal and external bisectors of the angle between AP BQ. The equation of the cylindroid is

$$z(x^2+y^2)-hxy \operatorname{cosec} \omega = 0.$$

The two principal pitches have opposite signs, and the given forces act along the two screws of zero pitch of this cylindroid.

236.] **Theorem.** When a system of forces is reduced to a pair of forces represented in magnitudes and lines of action by two right lines, the volume of the tetrahedron formed by these lines is constant, however the reduction is made.

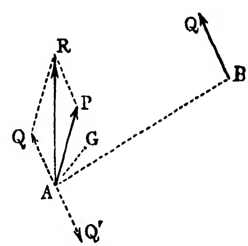

Fig. 241.

Let the system of forces be reduced to P and Q, and let these be supposed to act at the extremities, A and B, of the shortest distance between them. Now to get the force and couple corresponding to the origin A, introduce at this point two opposite forces, AQ and AQ', each equal and parallel to Q.

Compounding P and Q we get the resultant force, R; and taking the forces Q at B and Q' at A we get a couple whose axis, AG, is at right angles to the plane $QBAQ'$ and equal to $Q.AB$. Since AB is perpendicular to both P and Q, it is clear that AG is in the plane QAP and at right angles to AQ.

Now since (Art. 208) $G\cos\phi = K$, we have

$$Q.AB.\sin QAR = K.$$

But $\sin QAR = \dfrac{P}{R}.\sin PAQ$. Hence

$$P.Q.AB.\sin PAQ = K.R.$$

Now the volume of the tetrahedron formed by the lines AP and BQ

$= \frac{1}{3}$ area $ABQ \times$ perpendicular from P on the plane ABQ;

$= \frac{1}{6} BQ.AB \times AP.\sin PAQ$;

$= \frac{1}{6} P.Q.AB.\sin PAQ$.

Hence if Δ denotes the volume of the tetrahedron,

$$\Delta = \tfrac{1}{6} KR.$$

This theorem has been proved in various ways. For an elegant demonstration by Möbius, see *Crelle's Journal*, vol. iv,

p. 179, or Jullien's *Problèmes de Mécanique Rationnelle*, vol. i, p. 71.

237.] **Symmetrical Reduction of a System of Forces.** A system of forces can be reduced to two forces equal in magnitude, equally inclined at opposite sides to Poinsot's Axis, and equally distant from this axis.

Suppose the forces replaced by R acting along Poinsot's Axis, Oz, and a couple, K. Take any point, O' (Fig. 236); draw $O'O$ perpendicular to Oz and produce it to O'' so that $O'O = OO''$. Let R acting at O be replaced by $\frac{1}{2}R$ acting at O' and $\frac{1}{2}R$ acting at O''. Also let the forces of the couple act at O' and O''; for this purpose these forces must each be made $= \frac{1}{2}K/x$, where x is OO'. Now the resultant of $\frac{1}{2}R$ and $\frac{1}{2}K/x$ at O' is a force

$$= \frac{1}{2}\sqrt{R^2 + K^2/x^2},$$

acting towards the right, and the resultant of $\frac{1}{2}R$ and $\frac{1}{2}K/x$ at O'' is a force of the same magnitude acting towards the left of the figure.

If ω is the angle made with Poinsot's Axis by these new forces at O' and O'',

$$\tan\omega = K/Rx.$$

If we choose x so that $K/x = R\sqrt{3}$, each of the two symmetrical forces is equal to R, and they are inclined at an angle of 60° to Poinsot's Axis.

238.] **Analytical Condition for a Single Resultant.** We have just seen that a system of forces acting on a rigid body is, in general, equivalent to *two* forces. Let the forces be replaced by a single resultant force, R, acting at an arbitrary origin, O, and a couple G. Now the direction-cosines of R referred to axes Ox, Oy, and Oz, are (Art. 206),

$$\Sigma X/R,\ \Sigma Y/R,\ \Sigma Z/R;$$

and those of G are L/G, M/G, N/G.

Hence, if ϕ is the angle between G and R,

$$\cos\phi = \frac{L\Sigma X + M\Sigma Y + N\Sigma Z}{GR}. \tag{1}$$

Now if the resultant couple is in a plane containing R, one of its forces can be made to destroy R, and there will remain a single force; but if G and R are not at right angles to each

other, the system of forces cannot be equivalent to a single force. The required condition is, therefore, $\cos\phi = 0$, or

$$L\Sigma X + M\Sigma Y + N\Sigma Z = 0, \tag{2}$$

provided that ΣX, ΣY, *and* ΣZ *do not all vanish*; for if they do, R will also vanish, and ϕ will be illusory. In fact, in this case, since L, M, and N alone exist, the system of forces is equivalent to a couple.

239.] **Theorem.** The quantity $L\Sigma X + M\Sigma Y + N\Sigma Z$ has the same value for all systems of rectangular axes assumed anywhere in space.

From (1) of the last Art., it $= RG\cos\phi$, or RK, where K is Poinsot's moment (Art. 208).

Hence, if this quantity vanishes for any one set of axes, the force and the axis of the accompanying couple corresponding to any origin are at right angles.

The value of this quantity can be exhibited in another form, which also shows that it is independent of any particular set of axes.

Substituting for L, M, and N the values (Art. 206), $\Sigma(Zy - Yz)$, &c., the expression becomes

$$\begin{aligned}&(Z_1y_1 - Y_1z_1 + Z_2y_2 - Y_2z_2 + \dots)(X_1 + X_2 + \dots)\\ &+ (X_1z_1 - Z_1x_1 + X_2z_2 - Z_2x_2 + \dots)(Y_1 + Y_2 + \dots)\\ &+ (Y_1x_1 - X_1y_1 + Y_2x_2 - X_2y_2 + \dots)(Z_1 + Z_2 + \dots);\end{aligned}$$

or, substituting for X_1, Y_1, Z_1,... in terms of the forces P_1,... and their direction-cosines,

$$\begin{aligned}&[P_1(y_1\cos\gamma_1 - z_1\cos\beta_1) + P_1(y_2\cos\gamma_2 - z_2\cos\beta_2) + \dots]\\ &\qquad \times (P_1\cos\alpha_1 + P_2\cos\alpha_2 + \dots) + \&c. \dots\end{aligned}$$

It is clear at once that the terms P_1^2, P_2^2, ... disappear, and the products P_1P_2, P_1P_3, ... alone remain.

Collecting the coefficient of P_1P_2 as a typical term, we have

$$\begin{aligned}P_1P_2[&(x_1 - x_2)(\cos\beta_1\cos\gamma_2 - \cos\gamma_1\cos\beta_2)\\ &+ (y_1 - y_2)(\cos\gamma_1\cos\alpha_2 - \cos\alpha_1\cos\gamma_2)\\ &+ (z_1 - z_2)(\cos\alpha_1\cos\beta_2 - \cos\beta_1\cos\alpha_2)].\end{aligned}$$

Now (see Salmon's *Geometry of Three Dimensions*, or Frost's *Solid Geometry*) if (P_1, P_2) denotes the angle between the directions of the forces P_1 and P_2, the quantity in brackets

$= d_{12} . \sin(P_1, P_2)$, where d_{12} is the shortest distance between the lines of action of the forces.

Hence

$$L\Sigma X + M\Sigma Y + N\Sigma Z = \Sigma P_1 P_2 . d_{12} . \sin(P_1, P_2). \qquad (1)$$

Again (Art. 236), the term involving $P_1 P_2$ on the right side of (1) denotes six times the tetrahedron formed by P_1 and P_2; therefore the quantity on the left side is equal to *six times the sum (with their proper signs) of the* $\frac{1}{2}n(n-1)$ *tetrahedra which can be formed out of the pairs of lines representing the n forces*

$$P_1, P_2, \ldots, P_n.$$

This sum has, of course, no reference to any set of axes, and hence the necessarily *invariant* nature of $L\Sigma X + M\Sigma Y + N\Sigma Z$.

With regard to the sign to be given to any tetrahedron of the system, we define that—

The moment of a force with regard to a line is the component of the force perpendicular to the line multiplied by the shortest distance between the force and the line.

Hence $P_1 . d_{12} . \sin(P_1, P_2)$ is the moment of P_1 about the line of action of P_2. Now to determine the sign which must be given to any tetrahedron, let a watch be placed so that the direction in which either force acts passes perpendicularly from the back up through the face of the watch. If then the other force tends to produce rotation in the sense in which the hands rotate, the tetrahedron is to receive a negative sign, and if the rotation is the other way, a positive sign.

240.] **Conditions of Equilibrium of a Rigid Body acted on by any Forces.** The forces having been reduced to a resultant of translation, R, acting at any point, together with a corresponding couple, G, since a force and a couple cannot conjointly produce equilibrium ((ϵ), Art. 200) it is necessary that

$$R = 0 \text{ and } G = 0.$$

Substituting the values of R and G given in Art 206, we see that these two are equivalent to the following *six* conditions:

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$
$$L = 0, \quad M = 0, \quad N = 0,$$

which are the analytical expressions of the fact that *the forces must have no component along any line and no moment about any axis.*

241.] **Particular Cases of Equilibrium.** (*a*) Equilibrium of three forces. *When three forces keep a body in equilibrium, their lines of action must be coplanar and concurrent* (*or parallel*).

For, let the forces be P, Q, R. Then the sum of their moments about every right line $= 0$. Take any point, p, on P, and from it draw a line meeting Q—in q, suppose.

Since the sum of moments about the line pq must be zero, and since the moments of P and Q about it are separately zero, this line must intersect R—in r, suppose.

Draw another line through p meeting Q in q'; then, as before, this line must meet R—in r', suppose. Now, since two points on each of the lines Q and R lie in the plane determined by the lines pqr and $pq'r'$, the lines Q and R must each lie wholly in this plane. Again, draw any two lines whatever across Q and R: these must both be intersected by P; that is, P must lie in the plane of Q and R; hence all the forces are coplanar.

Finally, taking moments about the point (Art. 77) of intersection of Q and R, we see that P must pass through this point, and be equal and opposite to their resultant. If Q and R are parallel, P must be parallel to them, and equal and opposite to their resultant.

The case of Art. 19 is therefore the only case of equilibrium of three forces.

(*b*) Equilibrium of four forces. *If four forces keep a body in equilibrium, they must all lie on the same hyperboloid of one sheet.*

Any three non-intersecting right lines determine a hyperboloid of one sheet, because a surface of the second degree requires, in general, *nine* conditions for its determination, and the conditions that any one given right line ($x = az+m$, $y = bz+n$) should lie wholly on the surface are *three* in number; hence among the nine unknown coefficients in the equation of the surface there will be established nine (linear) equations if *three* given non-intersecting lines all lie on it. The surface is therefore determined.

Describe the hyperboloid determined by three of the forces, P, Q, R; then an infinite number of right lines can be drawn to intersect these three, and all such lines lie on the hyperboloid and constitute one system of its generators, while P, Q, R belong to the other system of generators (see Salmon's *Geometry of Three Dimensions*, Chap. VI). Every line intersecting P, Q, R

must, since the sum of the moments of the four forces about it $= 0$, also intersect the fourth force S; hence S passes through an infinite number of points lying on the hyperboloid, which is impossible unless S lies wholly on the surface.

The given forces, therefore, act along lines which are all generators of the same system of the same hyperboloid.

(c) Equilibrium of five forces. *If five forces keep a body in equilibrium, their lines of action must intersect two right lines.* If a right line could be drawn so as to intersect four of the forces, it would have to intersect the fifth, on account of the vanishing of the sum of the moments about it.

Now *two* right lines can, in general, be drawn to intersect any four non-intersecting right lines. For, let the four lines be denoted by P, Q, R, S. Construct the hyperboloid determined by (P, Q, R), and also the hyperboloid determined by (P, Q, S). These hyperboloids having two right lines for a part of their curve of intersection will have two other right lines for the remainder of the curve. For, let the equations of the line P be $(x = 0,\ y = 0)$, and those of Q be $(z = 0,\ w = 0)$; then the equation of any hyperboloid containing these lines is

$$x\,(mz + pw) + y\,(lz + qw) = 0\,;$$

another hyperboloid containing the same lines is

$$x\,(m'z + p'w) + y\,(l'z + q'w) = 0.$$

Now at all points of intersection of these two hyperboloids, for which x and y do not both vanish, and for which z and w do not both vanish—i. e. at all points of their curve of intersection, excluding the points on the two common generators—we have

$$\frac{mz + pw}{m'z - p'w} = \frac{lz + qw}{l'z + q'w}.$$

This equation, being homogeneous in z and w, denotes two planes passing through the line Q; but any plane through a generator must intersect the surface again in a right line; therefore these two planes cut the surface in two right lines, which are the remaining part of the curve of intersection of the two hyperboloids; and each of them intersects the generators (P, Q, R) and the generators (P, Q, S); i. e. each intersects the four lines P, Q, R, S. Each must, therefore, intersect the fifth force. Q.E.D.

EXAMPLES.

1. A rigid body is acted on by forces represented in magnitudes and lines of action by the sides of a gauche polygon taken in order; prove that the forces are equivalent to a couple, and that the sum of their moments about any line is represented by double the area of the projection of the polygon on a plane perpendicular to the line.

Let the forces be represented by the lines AB, BC, CD, ...(Fig. 242), and let OQ be any axis.

On the axis take any point, O, and reduce the forces to a resultant, R, of translation at this point, together with a couple, G (Art. 206). This is done by introducing at O two forces parallel and equal to AB in opposed directions, two equal and opposite to BC, &c. Now (Art. 199) the resultant of translation vanishes, and the component couples are represented by double the areas of the triangles OAB, OBC, &c. If the axes of these couples are drawn at O, the sum of the moments of the forces about OQ will be represented by the sum of the components of the axes along OQ; but this is the same as double the sum of the projections of the areas of the triangles on a plane perpendicular to OQ; that is, the moment about OQ is represented by double the area of the projection of the polygon on a plane perpendicular to OQ.

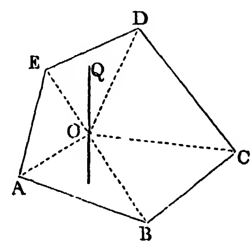

Fig. 242.

Again, since G is the greatest moment round any axis through O (Art. 206), it follows that the axis of the resultant couple is the line perpendicular to the plane on which the projected area of the polygon is a maximum.

2. When the resultant of translation vanishes, the forces will be in complete equilibrium if the sums of their moments round any three non-coplanar axes are separately equal to nothing.

For if L is the moment round the axis of x, the moment L', round a parallel axis through the point (α, β, γ) is $L+\gamma\Sigma Y-\beta\Sigma Z$. Hence $L'=L$, $M'=M$, $N'=N$; and since the moment round an axis through (α, β, γ) making angles λ, μ, ν with the axes of co-ordinates is $L'\cos\lambda+M'\cos\mu+N'\cos\nu$, it follows that the moments round all parallel axes are equal. For the three axes of moments we may take, therefore, three lines through the origin making angles $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, and $(\lambda_3, \mu_3, \nu_3)$ with the axes of co-ordinates. Suppose then that

$$L\cos\lambda_1+M\cos\mu_1+N\cos\nu_1=0,$$
$$L\cos\lambda_2+M\cos\mu_2+N\cos\nu_2=0,$$
and
$$L\cos\lambda_3+M\cos\mu_3+N\cos\nu_3=0.$$

These require either that $L = M = N = 0$, or that

$$\begin{vmatrix} \cos\lambda_1, & \cos\mu_1, & \cos\nu_1 \\ \cos\lambda_2, & \cos\mu_2, & \cos\nu_2 \\ \cos\lambda_3, & \cos\mu_3, & \cos\nu_3 \end{vmatrix} = 0.$$

The latter condition requires that the three axes of moments be in one plane. If they are not coplanar, we must have $L = M = N = 0$, i.e. the forces are in equilibrium.

3. A tetrahedron is acted on by forces applied perpendicularly to the faces at their respective centroids. If the force applied to each face is proportional to the area of that face, prove that the tetrahedron is in equilibrium, the forces being supposed to act all inwards or all outwards.

Let A, B, C, D be the vertices of the tetrahedron, and denote the areas of the faces opposite these vertices by A_1, B_1, C_1, D_1, respectively. Denote also the angle between the faces A_1 and B_1 by $\widehat{A_1B_1}$. Then evidently

$$A_1 = B_1 \cos\widehat{A_1B_1} + C_1 \cos\widehat{A_1C_1} + D_1 \cos\widehat{A_1D_1};$$

or, if the forces perpendicular to the faces are denoted by P, Q, R, S,

$$P - Q \,.\cos\widehat{PQ} - R \,.\cos\widehat{PR} - S \,.\cos\widehat{PS} = 0,$$

which shows that there is no resultant force in a direction perpendicular to the face A_1; similarly there is no resultant force in directions perpendicular to the other faces; therefore the resultant of translation vanishes.

To show that there is no resultant couple, let each force be replaced by three equal forces acting at the vertices of the corresponding face. Thus the force P is to be replaced by three forces each equal to $\frac{1}{3}P$ acting at the points B, C, D perpendicularly to the face BCD. Let us calculate the sum of the moments of the forces about the edge BC. For this purpose, let the forces $\frac{1}{3}Q$ and $\frac{1}{3}R$ at D be each resolved in the direction of the force $\frac{1}{3}P$ at this point, i.e. perpendicularly to the face BCD. Supposing the forces to act outwards, the components of $\frac{1}{3}Q$ and $\frac{1}{3}R$ are $-\frac{1}{3}Q\,.\cos\widehat{PQ}$ and $-\frac{1}{3}R\,.\cos\widehat{PR}$; therefore the sum of the moments of the forces at D about BC is proportional to

$$(A_1 - B_1 \,.\cos\widehat{A_1B_1} - C_1 \,.\cos\widehat{A_1C_1})\,p',$$

or $$D_1 \,.\, p' \,.\cos\widehat{A_1D_1},$$

or, again, $$D_1 \,.\, p \,.\cot\widehat{A_1D_1},$$

p' being the perpendicular from D on BC, and p the perpendicular from D on the base ABC. But this last expression is three times the volume of the tetrahedron multiplied by $\cot\widehat{A_1D_1}$. In the same way, the sum of the moments of the forces at A is represented by three times the volume of the tetrahedron multiplied by $\cot\widehat{A_1D_1}$; and as these moments are in opposite senses, the forces have no moment round

the edge BC, and similarly no moment round any of the edges. Hence by the last example they are in equilibrium.

For another simple method of proof see Collignon's *Statique*, p. 354.

4. Prove that a solid body of any shape is in equilibrium if it is acted on throughout its surface by normal forces, each force being proportional to the superficial element on which it acts.

One very simple method of proof consists in imagining a surface precisely equal and similar to that of the given body to be traced out in a weightless fluid which is subject to any pressure.

5. If a curved surface whose edge is a plane curve is acted on all over its surface by normal forces, each proportional to the element of surface on which it acts, prove that these forces have a single resultant if they all act towards the same side of the surface.

6. Forces perpendicular and proportional to the areas of the faces act at the centres of the circles circumscribing the faces of a tetrahedron; prove that they are in equilibrium, if they all act inwards or outwards.

They meet in the centre of the circumscribed sphere. The proposition is evidently true also for any polyhedron bounded by triangular faces.

Taking the results of this example and Example 3 together, we see that forces proportional to the areas and perpendicular to them are in equilibrium if they act at the orthocentres of the triangular faces of any polyhedron.

7. Find the force necessary to keep a heavy door in a given position, the hinge line being inclined to the vertical and the hinges smooth.

Let i be the inclination of the hinge line to the vertical, and a the given inclination of the plane of the door to the vertical plane containing the hinge line. Then if W is the weight of the door, a the distance of its centre of gravity from the hinge line, and θ the angle between the normal to the plane of the door and the vertical, the moment of the weight about the hinge line is

$$Wa \cos \theta.$$

This is the moment of the required force. To find θ, let lines parallel to the hinge line and the vertical be drawn through any point, O, and through this point let a plane be drawn parallel to the plane of the door. Round O let any sphere be described; let V and L (Fig. 243) be the points where these lines meet the sphere; DL the circle in which the plane of the door intersects the sphere, and N the point in which the normal, ON, to the door intersects it. Then $VL = i$, $\angle DLV = a$, and $NV = \theta$, and we have from the spherical triangle VDL,

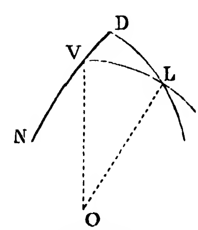

Fig. 243.

$$\sin VD = \sin i \sin a,$$

or

$$\cos \theta = \sin i \sin a,$$

since N is the pole of DL. Hence the moment of the required force is

$$Wa \sin i \sin a,$$

and when its point of application and direction are known, its magnitude is therefore known.

8. A beam can turn in every direction about one end which is fixed; the other end rests on a rough inclined plane. Find the limiting position of equilibrium. (See Walton's *Mechanical Problems*, p. 191, third edition.)

Let AB (Fig. 244) be the beam, A the fixed end, DPH the rough inclined plane, PH the intersection of this plane with a horizontal plane through A, APD the vertical plane through A perpendicular to the inclined plane, BD a line parallel to PH, AO a perpendicular from A on the inclined plane, DQ a perpendicular on the horizontal plane, i the inclination of the plane, α the angle, ABO, between the beam and this plane, and μ the coefficient of friction.

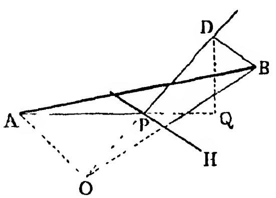

Fig. 244.

Now *suppose first that the beam is perfectly inelastic.* Then the end B describes on the inclined plane a circle whose centre is O, and if it is about to slip, the force of friction assumes a direction perpendicular to OB in the inclined plane. The extreme position of the beam will be denoted by the angle, θ or DOB, between the plane, AOB, through the beam normal to the inclined plane and the vertical plane, AOD.

The forces acting on the beam are its weight, the reaction of the smooth joint at A, and the total resistance of the inclined plane at B. This last force we shall consider as composed of a normal reaction, R, and a force of friction, μR, acting perpendicularly to BO. For the equilibrium of the beam take moments about a vertical axis through A. The moment of the normal reaction at B is $R \sin i \times BD$, or $R \sin i \,.\, BO \sin \theta$, or again, $R \sin i \,.\, AB \cos \alpha \sin \theta$. To find the moment of μR, resolve it into $\mu R \cos \theta$ along BD and $\mu R \sin \theta$ parallel to OD; and resolve this latter again into a horizontal component, $\mu R \sin \theta \cos i$, and a vertical component, $\mu R \sin \theta \sin i$. The moment of μR is then equal to the sum of the moments of $\mu R \cos \theta$ and $\mu R \sin \theta \cos i$; that is, it is equal to

$$\mu R \cos \theta \times AQ + \mu R \sin \theta \cos i \times BD.$$

Hence the equation of moments is

$$R (\sin i - \mu \cos i \sin \theta) BD = \mu R \cos \theta \,.\, AQ.$$

But $$AQ = AP + PQ = \frac{AO}{\sin i} + (OD - OP) \cos i$$

$$= \frac{AB \,.\, \sin \alpha}{\sin i} + AB \cos i \cos \alpha \cos \theta - AB \sin \alpha \cot i \cos i$$

$$= AB (\sin i \sin \alpha + \cos i \cos \alpha \cos \theta);$$

therefore

$$(\sin i-\mu\cos i\sin\theta)\cos\alpha\sin\theta = \mu\cos\theta\,(\sin i\sin\alpha+\cos i\cos\alpha\cos\theta),$$

or $$\sin i\cos\alpha\sin\theta = \mu\cos i\cos\alpha+\mu\sin i\sin\alpha\cos\theta,$$

or $$\sin\theta-\mu\tan\alpha\cos\theta = \mu\cot i.$$

Putting $\mu\tan\alpha = \tan\beta$, we have θ from the equation

$$\sin(\theta-\beta) = \mu\cot i\cos\beta. \tag{1}$$

If there is no horizontal plane through A obstructing the beam, it will be possible for the end B to describe a complete circle round O. Let us inquire the condition that the beam should rest in all possible positions. For this there must be no *limiting* position of equilibrium, or, in other words, the value of θ in (1) must be imaginary.

The required condition is, then, $\mu\cot i\cos\beta > 1$,

that is, $$\mu > \frac{\tan i}{\sqrt{1-\tan^2 i\tan^2\alpha}}.$$

Let us next *suppose that the beam is elastic*, or that, in virtue of a compression of the beam, B is not constrained to move in the circle whose centre is O. Supposing, then, that the beam has been jammed against the plane, if the coefficient of friction is gradually diminished, B will begin to move in some other direction than that perpendicular to OB, and this direction will be exactly opposite to that in which the force of friction acts. Now the reaction at A, the total resistance at B, and the weight of the beam lie in one plane which must, therefore, be *the vertical plane through the beam.* The total resistance at B must, moreover, lie inside or on the cone of friction described round B. Hence if the position of the beam is such that the vertical plane through it *touches* this cone, equilibrium will be at its limit, since the line of action of the total resistance is the line of contact of the vertical plane with the cone.

Let the lines and planes of the figure be projected on a sphere described about B as centre with arbitrary radius. Then the cone of friction will appear as a small circle of angular radius, NC (Fig. 245), equal to λ, the angle of friction. Let N be the point in which the normal to the inclined plane at B meets the sphere ; A, the point representing the beam, and ACV the vertical plane through the beam touching the cone of friction. Now the vertical line at B lies in the vertical plane, ACV, through the beam, and it makes an angle equal to i with the normal to the inclined plane. Hence, take a point V in ACV so that $NV = i$, and we have NV, the circle answering to the vertical plane through B normal to the inclined plane (a plane

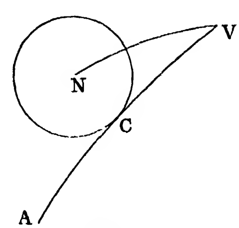

Fig. 245.

which is parallel to the plane APD, Fig. 244). In the spherical triangle NVC we have, then,

$$\sin NV . \sin NVC = \sin NC,$$

or
$$\sin i \sin \theta = \sin \lambda;$$

$$\therefore \quad \sin \theta = \frac{\sin \lambda}{\sin i}.$$

This second solution supposes that *the only condition to which the total resistance is subject is that of making with the normal an angle not greater than the angle of friction.* The supposition of perfect rigidity, on the contrary, restricts the direction of the force of friction in the inclined plane, making it perpendicular to the line OB.

9. A heavy elastic beam rests on two rough inclined planes whose intersection is a horizontal line. Show that every position of the beam may be one of equilibrium if the inclination of each plane is less than the angle of friction for that plane and the beam.

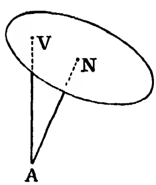

Fig. 246.

Let A (Fig. 246) be one end of the beam, AN the normal to the plane on which A rests, and AV the vertical at A. Then if the beam is sufficiently elastic, it may be jammed against the planes, and the only condition to which the total resistances at its ends are subject are the conditions of making with the normals angles not greater than the corresponding angles of friction. Hence in the extreme position in which the end A is about to slip, the vertical plane through the beam must touch the cone of friction described round the normal, AN. But this is manifestly impossible, since the angle λ is $> VAN$; for the vertical line is included within the cone, and through this line no plane can be drawn to touch the cone. There can, therefore, be no *limiting* equilibrium at either end in any position of the beam.

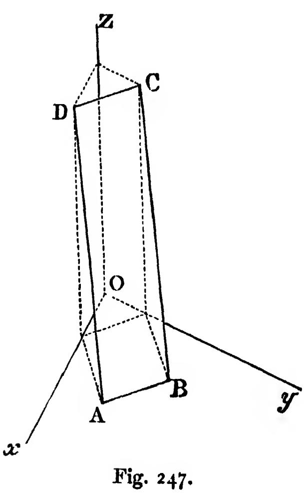

Fig. 247.

10. A ladder, $ABCD$ (Fig. 247), whose centre of gravity divides it into two given segments, rests with one end, AB, on the ground, the upper end, CD, resting symmetrically against two equally rough vertical planes which include a given angle; find its limiting inclination to the ground.

On account of the equal roughness of the vertical walls and the symmetrical position of the ladder, the total resistances at C and D are equal; moreover they have a single resultant passing through the middle point of CD, since the two normal

pressures and the two forces of friction have resultants passing through this point.

At each of the points A and B the total resistance makes the angle of friction with the normal, and the resultant of these forces acts at the middle point of AB, making the angle of friction with the vertical. The resultant resistance above and that below must meet in the vertical through the centre of gravity of the ladder.

Let λ be the angle of friction at the ground; $\lambda' =$ that for each wall; $a =$ lower and $b =$ upper segment of ladder made by its centre of gravity; $\theta =$ limiting inclination of ladder; $\phi =$ angle made with vertical by the resultant of the total resistances at C and D. Then, by the 'cotangent formula' of Art. 35, we have

$$(a+b)\tan\theta = \frac{a}{\mu} - b\cot\phi, \tag{1}$$

where $\mu = \tan\lambda$.

The angle ϕ may, of course, be found by the ordinary method of determining the magnitude and direction of the resultant of forces from their several components; but we prefer to employ for the purpose the *method of spherical projection*, which is more simple, and which will be frequently employed in the sequel. The method consists in constructing a sphere of any radius, and drawing through its centre lines and planes parallel to the lines and planes in our figure; these will intersect the surface of the sphere in points and circles, respectively,—as illustrated in Examples 7, 8, 9 already.

Let O (Fig. 248) be the centre of the sphere; OZ a parallel to the vertical; ON and ON' parallels to the normals to the planes zy and zx, respectively; ZC and ZD planes parallel to these planes respectively; OR and OR' lines in the planes ZN and ZN', each inclined at the angle, λ', of friction to the corresponding normal; then OR and OR' represent the lines of action of the total resistances at C and D. If S is the middle point of the arc RR', the resultant of the resistances acts in OS, and the arc $ZS = \phi$. If α is the angle between the walls, $DC = \alpha$, and $NN' = \pi - \alpha$; therefore the angle $RZS = \frac{1}{2}\pi - \frac{1}{2}\alpha$; and applying Napier's Analogies to the triangle RZS, we have

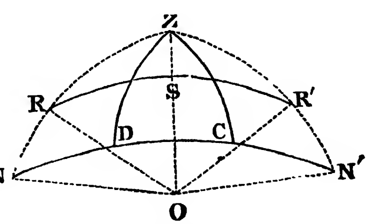

Fig. 248.

$$\cot\phi = \mu' \operatorname{cosec}\tfrac{1}{2}\alpha.$$

Hence (1) gives

$$(a+b)\tan\theta = \frac{\alpha}{\mu} - b\mu' \operatorname{cosec}\tfrac{1}{2}\alpha, \tag{2}$$

which determines the limiting inclination.

11. If the vertical walls are unequally rough, show that the initial motion of the ladder cannot be one in which the line CD moves down parallel to its original position.

12. If the walls are unequally rough, show that the initial motion cannot be one in which one corner (D) is for the moment at rest, while slipping takes place at two other corners (C and B).

13. A solid rectangular block is placed with one of its faces on an inclined plane so rough as to prevent slipping, while tumbling is possible; to investigate the positions of equilibrium.

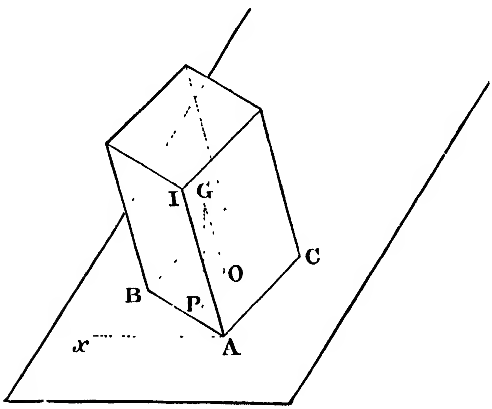

Fig. 249.

Let ABC (Fig. 249) be the face on the inclined plane. All the different positions may be obtained by turning the block round the edge, AI, through any corner of the base, which is perpendicular to the inclined plane. Draw the horizontal line Ax in the inclined plane. Let G be the centre of gravity of the block; O that of the face ABC; GP the vertical line through G meeting the face ABC in P. Since GO is perpendicular to the inclined plane, $\angle PGO = i =$ inclination of plane, so that the sides of the triangle OGP are all constant whatever be the position of the block; therefore if the successive positions of P are marked on the face ABC, they trace out in it a circle with centre O.

Again, since Ax is the line of intersection of the inclined plane and a horizontal plane, it is at right angles to the plane of two intersecting normals to these planes; it is therefore at right angles to the plane of GO and GP, and hence to OP in all positions of the block.

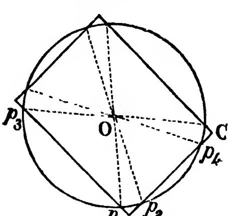

Fig. 250.

Therefore if Fig. 250 represents the base of the block and the circle traced out in it by the motion of P, the points in which the circle intersects the sides of the face being p_1, p_2, p_3, p_4, we see that if we turn the block round AI so that any one of the lines Op_1, Op_2, Op_3, Op_4 is at right angles to Ax, we obtain a position in which the block is about to tumble; in other words, make a perpendicular to any of the lines $Op_1, Op_2, \ldots$

(drawn in the plane of the base) horizontal, and we obtain a limiting position.

If $2a$ and $2b$ are the lengths of the edges AB and AC, and c is the distance of G from the base, the conditions that the circle should intersect all the edges of the base are $c \tan i > a$ and $c \tan i > b$, where i is the inclination of the plane.

Obviously if the block is any solid body having a base of any form, the solution is the same. If O is the projection of G on the base, and $OG = c$, describe a circle round O with radius $c \tan i$; let p be any point of intersection of this circle with the contour of the base; then make a perpendicular to Op horizontal, and we obtain a limiting position.

14. A heavy uniform bar rests with its extremities on two rough inclined planes whose line of intersection is horizontal; supposing that the bar is slightly elastic and can be jammed between the planes, investigate its positions of limiting equilibrium.

We may evidently consider the centre of the bar to be restricted to a fixed vertical plane which is perpendicular to both of the inclined planes. Take this plane as that of yz, the axis of x being the line of intersection of the inclined planes, and the axis of z a vertical line. Let (y, z) be the co-ordinates of the centre of gravity of the bar; $2a$ = length of the bar; θ = angle between the bar and a vertical line; ϕ = angle between vertical plane through the bar and the plane yz; i and i' the inclinations of the given planes; λ and λ' the angles of friction between them, respectively, and the bar.

Then the co-ordinates of the extremities of the bar are

$a \sin\theta \sin\phi$; $y + a \sin\theta \cos\phi$; $z + a\cos\theta$ for one extremity, A,

$-a \sin\theta \sin\phi$; $y - a \sin\theta \cos\phi$; $z - a\cos\theta$ for the other, B.

Since these lie on the inclined planes, we have

$$z + a\cos\theta = (y + a\sin\theta\cos\phi)\tan i, \tag{1}$$

$$z - a\cos\theta = (y - a\sin\theta\cos\phi)\tan i'. \tag{2}$$

Now, as in Example 9, if the first end is going to slip,

$$\sin i \sin\phi = \sin\lambda, \tag{3}$$

since the vertical plane through the beam touches the cone of friction at this end. If the other end were about to slip, we should have

$$\sin i' \sin\phi = \sin\lambda'; \tag{4}$$

so that both ends cannot slip at once unless

$$\frac{\sin\lambda}{\sin i} = \frac{\sin\lambda'}{\sin i'}.$$

Let t and t' stand for $\tan i$ and $\tan i'$; then, eliminating θ from (1) and (2), we have

$$[2z - (t-t')y]^2 \sec^2\phi + [(t-t')z + 2tt'y]^2 = a^2(t+t')^2.$$

Hence the positions in which either end is about to slip are such

that the centre of gravity lies on a certain ellipse—any position of this point on the ellipse being admissible—the corresponding value of ϕ being given by (3) or (4), and that of θ by (1) or (2).

We have now to determine, however, whether *both* ellipses are admissible or not—i. e. whether there are positions in which the end A is about to slip, while B remains at rest, and also positions in which B is about to slip while A remains at rest.

Assuming that A is about to slip, the vertical plane through the bar touches the cone of friction described around the normal at A to the inclined plane (i); but at the same time this vertical plane must not lie wholly outside the cone of friction at B, i.e. it must intersect this latter in two real right lines. Now if, for simplicity, we transfer the origin to B, the axes remaining unchanged in direction, the equation of the vertical plane through the bar is

$$x - y\tan\phi = 0,$$

and the cone of friction at B is

$$(y\sin i' + z\cos i')^2 - \cos^2\lambda'(x^2+y^2+z^2) = 0;$$

and these will intersect in a pair of real lines if

$$\sin i'\sin\phi < \sin\lambda',$$

or by (3),

$$\frac{\sin\lambda}{\sin i} < \frac{\sin\lambda'}{\sin i'}.$$

If this inequality is satisfied, it is only the end A that can slip; if the reverse holds, it is the end B that can slip. Thus both values of ϕ are not admissible.

15. If at any point, P, a plane, ϖ, is drawn perpendicular to the axis of principal moment at the point, find the envelope of ϖ as P moves along a given curve.

Simplicity will be gained by taking Poinsot's Axis, Oz (Fig. 236), as axis of z. Let (α, β, γ) be the co-ordinates of P with reference to Oz and any two axes of x and y. Then, introducing two forces equal and opposite to R at P, we shall have the whole force system equivalent to R at P, Poinsot's couple K, and a couple $R\rho$, where ρ is the perpendicular from P on Oz. We may replace the couple $R\rho$ by two components parallel to Ox and Oy, and these will be $-R\beta$ and $R\alpha$; so that the component axes of the principal couple G at P are $(-R\beta, R\alpha, K)$. Hence the equation of the plane ϖ is

$$-R\beta(x-\alpha) + R\alpha(y-\beta) + K(z-\gamma) = 0,$$

or

$$\beta x - \alpha y - \frac{K}{R}(z-\gamma) = 0. \tag{1}$$

If the equations of the curve along which P moves are

$$\alpha = \phi(\gamma), \quad \beta = \psi(\gamma),$$

substitute these values of α and β in (1), and eliminate γ from the resulting equation and its derived with respect to γ.

Verify, in particular, the result of Art. 235, that if P moves in a right line, ϖ will turn round another right line; and that Poinsot's Axis

intersects the shortest distance between these two lines, dividing it in the ratio $\frac{\cot\theta}{\cot\phi}$ (Art. 212).

16. Find the surface traced out by the axes of principal moment at points taken along a right line intersecting Poinsot's Axis perpendicularly.

Let Ox (Fig. 236) be the assumed line, and let it be taken as axis of x, Poinsot's Axis, OK, being that of z. Let $OO' = x$, and let (y, z) be the co-ordinates of any point on $O'G$. Then, if $\phi = \angle GO'K'$, we have

$$\frac{z}{y} = \cot\phi = \frac{Gn}{O'n} = \frac{K}{Rx},$$

or

$$xz = \frac{K}{R}.y,$$

an equation which denotes a hyperbolic paraboloid. As the point O' moves out from O along Ox, the axes (such as $O'G$) of principal moment revolve towards the right; as O' moves in towards O, they revolve towards the left, and, after coincidence with Poinsot's Axis at O, they still revolve towards the left. At an infinite distance from O they are at right angles to Poinsot's Axis.

17. Find the surface traced out by the axes of principal moment at points taken all along any arbitrary curve.

From Example 15, the equations of the principal axis at the point (α, β, γ) with reference to Poinsot's Axis as axis of z, and any two rectangular axes of x and y are

$$\frac{x-\alpha}{-\beta} = \frac{y-\beta}{\alpha} = \frac{z-\gamma}{p},$$

where p is the pitch of the wrench to which the given forces are equivalent. From these we have

$$\alpha = p.\frac{px+y(z-\gamma)}{(z-\gamma)^2+p^2}; \quad \beta = p.\frac{py-x(z-\gamma)}{(z-\gamma)^2+p^2};$$

and if the point (α, β, γ) moves along the curve whose equations are

$$\phi(\alpha, \beta, \gamma) = 0, \quad \psi(\alpha, \beta, \gamma) = 0,$$

substitute the above values of α and β in these equations and then eliminate γ. The resulting equation in x, y, z is that of the surface traced out.

18. A plank, AB, laid on a rough inclined plane, has attached to its upper extremity, A, a cord which lies along the plane in the direction of the plank and is pulled with a constant force, P; find the limiting position of equilibrium of the plank.

Ans. Let W = weight of plank, i = inclination of the plane, λ = angle of friction, and θ = inclination of the plank to a horizontal line drawn in the inclined plane; then

$$\sin\theta = \frac{P^2 + W^2(1-\cos^2 i \sec^2\lambda)}{2PW\sin i}.$$

19. Show that the initial motion of the plank will be one of translation simply, in a direction making with a horizontal line in the inclined plane an angle ϕ determined by the equation

$$\tan\phi = \frac{P\sin\theta - W\sin i}{P\cos\theta},$$

where θ has the value found in last example.

20. If $P = 0$, explain the values of θ in the cases

$$i > \lambda, \quad i < \lambda, \quad i = \lambda.$$

21. Find the value of P so that the direction of slipping shall be at right angles to the direction of the plank, and find θ.

Ans. $P = W\sqrt{1-\cos^2 i \sec^2\lambda}$, and $\cos\theta = \dfrac{\tan\lambda}{\tan i}$.

[This case is the same as that in which the cord is replaced by a smooth pivot at the extremity A.]

22. A triangular prism is placed with its triangular face on a rough inclined plane, which is rough enough to prevent slipping; find the greatest height of the prism so that there may be at least one position of equilibrium.

Ans. If i = inclination of plane, and if the sides of the triangular face are a, b, c, in descending order of magnitude, the greatest height is

$$\tfrac{2}{3}\sqrt{2a^2+2b^2-c^2}.\cot i.$$

23. A heavy plate of any form rests on two rough fixed pegs A and B, the line joining which is not horizontal; the plate can turn round a pivot, without friction, at a point C; if C is raised so that the plate turns gradually about the fixed line AB, find the inclination of the plane ABC to the horizon when the plate begins to slip on the pegs.

24. A particle is acted on by any number of given forces, P_1, P_2, ...; prove that if R is their resultant,

$$R^2 = \Sigma(P^2) + 2\Sigma(P_1.P_2\cos\widehat{P_1P_2}),$$

where $\widehat{P_1P_2}$ denotes the angle between the directions of P_1 and P_2.

25. Prove that a system of forces acting on a rigid body may be replaced by two equal forces whose lines of action are perpendicular to each other, and each inclined at an angle of 45° to Poinsot's Axis: the forces act at the ends of a line bisected by this axis; the length of this line is $\dfrac{2K}{R}$, and each force is $\dfrac{R}{\sqrt{2}}$, R being the resultant of translation, and K Poinsot's moment.

26. Prove that the distance between the lines of action of the two rectangular forces which equivalently replace a given system of forces is a minimum when the forces are equal.

27. $ABCD$ is a tetrahedron; forces P, Q, R act along the edges BC, CA, AB in order, and forces P', Q', R' act along AD, BD, CD; prove that the condition for a single resultant is

$$\frac{PP'}{BC.AD}+\frac{QQ'}{CA.BD}+\frac{RR'}{AB.CD}=0.$$

28. A rough heavy body, bounded by a curved surface, rests upon two others which themselves rest on a rough horizontal plane; show that the three centres of gravity and the four points of contact lie in one plane.

29. A heavy beam rests on two smooth inclined planes; show that their line of intersection must be perpendicular to the beam and parallel to the horizon.

30. Prove that the moment of a force represented by the right line PQ about a right line AB is six times the volume of the tetrahedron $ABPQ$ divided by AB.

31. Three equal heavy spheres hang in contact from a fixed point by strings of equal length; find the weight of a sphere of given radius which when placed upon the other three will just cause them to separate.

Ans. If W and a are the weight and radius of each of the three spheres, W' and r the weight and radius of the superincumbent sphere, and l the length of each string,

$$\frac{W'}{W'+3W}=\sqrt{\frac{3r^2+6ar-a^2}{3l^2+6al-a^2}}.$$

32. Three spheres are placed in contact on a rough horizontal plane, and a fourth sphere is placed upon them, there being no friction between the spheres themselves. Show that equilibrium is impossible.

33. Three equal spheres are placed in contact on a rough horizontal plane, and a fourth sphere is placed upon them, there being friction between the spheres themselves. Find the least coefficient of friction between the spheres which will allow of equilibrium.

Ans. If a is the radius of each of the equal spheres and r that of the superincumbent sphere, the least value of λ, the angle of friction, is given by the equation

$$\sin 2\lambda=\frac{2}{\sqrt{3}}.\frac{a}{a+r}.$$

(The total resistance between the upper sphere and any one of the lower spheres must be capable of acting through the point of contact of the latter and the ground.)

34. Three forces whose lines of action are given, but not their magnitudes, have a single resultant. Prove that the surface traced out by the line of action of the resultant is a hyperboloid of one sheet.

(Draw any three lines across the given lines of action. Then the line of action of the resultant must always intersect these three.)

35. A heavy triangular plate of uniform thickness is suspended from a fixed point by means of three strings attached to the point and to the vertices of the plate; prove that the tension in each string is proportional to the length of the string.

(Let O be the fixed point, A, B, C the vertices of the plate, and G its centre of gravity.

Then G must lie vertically under O. Take $3\,OG$ to represent the weight of the plate. Then, by Leibnitz's graphic representation [Art. 199], the force $3\,OG$ may be resolved into the forces OA, OB, OC. But a given force can have only one set of components along three given concurrent lines. Therefore, &c.)

36. At points on any right line the axes of principal moment of a given system of forces are drawn; prove that their extremities trace out another right line. (Wolstenholme's *Problems.*)

(At any point O on the given line draw R and G. Take as axes of x, y, and z the given line, the line OG, and a line at O perpendicular to R and the given line. Then if at any point P on the given line at a distance x from O the axis of principal moment is drawn, the co-ordinates of its extremity will be x, G, and $Rx\sin\alpha$, where α is the angle which R makes with the given line. Hence the extremities lie on the line $y = G$, $z = Rx\sin\alpha$.)

37. Prove that the axes of principal moment at points along any right line whatever trace out a hyperbolic paraboloid.

(With the same axes as in last example, the surface has for equation $xy = \dfrac{G}{R\sin\alpha}\cdot z$.)

38. Find the condition that a given right line should intersect Poinsot's Axis.

Ans. If the equations of the line are $x = mz+p$, $y = nz+q$, the required condition is

$$R\,[mL+nM+N+q(X-mZ)-p(Y-nZ)] = K(mX+nY+Z),$$

where X is used for ΣX, &c.

(It will be found that the equations of Poinsot's Axis can be put into the forms

$$x = \frac{X}{Z}z+\frac{KY-MR}{RZ}, \quad y = \frac{Y}{Z}z-\frac{KX-LR}{RZ},$$

the origin being anywhere.)

39. A given system of forces is to be reduced to two inclined at the angle α; prove that the shortest distance between their lines of action cannot be less than $2G/R\tan\frac{1}{2}\alpha$. (Wolstenholme's *Problems.*)

40. Given any system of forces, find the point on a given right line at which the axis of principal moment is least inclined to the line.

Ans. The foot of the shortest distance between Poinsot's Axis and the given line.

[Most easily seen by spherical projection. Let O be any point on the given right line; round O as centre describe a sphere of any radius; let the given right line, OL, cut the sphere in L; let the resultant of translation at O and the axis, G, of principal moment at O cut the sphere in R and G, respectively. Draw the great circle arcs LR, LG. Then at any distance, x, along OL from O, the axis of principal moment is the resultant of an axis equal and parallel to G, and an axis Rx perpendicular to the plane LOR. Let a line, OQ, drawn through O parallel to this latter meet the sphere in Q. Draw the great circle arc, QG, meeting LR in H, suppose. Then the resultant of G and Rx is an axis somewhere in the plane QG; but, Q being the pole of LR, the arc LH is perpendicular to QG, and therefore is the least arc that can be drawn from L to QG. Hence when Rx and G give a resultant along OH, the axis of principal moment is least inclined to OL. Poinsot's centre being always sought on a line perpendicular to R and to the axis of principal moment at any point, the rest follows.]

41. The first case considered in Example 8 is, equally with the second, a geometrico-statical problem. Solve it without any mention of force.

(Express the condition that the vertical through the extremity A of AB is intersected by a line inclined at the angle λ to the normal at B, this line lying in the plane of the normal and a perpendicular to OB.)

42. AQB is any unclosed curve in space, A and B its extremities, and Q any variable point on the curve; P is any fixed point in space, $PQ = r$, ds = element of length at Q, θ = angle between PQ and ds. If each element ds is acted upon by a force $k\dfrac{\sin\theta \, . \, ds}{r^2}$ perpendicular to the plane of PQ and ds, k being a constant, find the resultant force and couple of this force system.

Ans. Let (α, β, γ), (x_1, y_1, z_1), (x_2, y_2, z_2) be the co-ordinates of P, A, B; let $PA = r_1$, $PB = r_2$, L, M, N the component moments round axes through P; and let

$$\int\frac{dx}{r} = F, \int\frac{dy}{r} = G, \int\frac{dz}{r} = H.$$

Then

$$X = k\left(\frac{dG}{d\gamma} - \frac{dH}{d\beta}\right), \text{ with similar values of } Y, Z;$$

$$L = k\left(\frac{\alpha - x_2}{r_2} - \frac{\alpha - x_1}{r_1}\right), \text{ with similar values of } M, N.$$

The axis of resultant moment is the external bisector of BPA, and $= k \sin \frac{1}{2} BPA$.

Hence, if the curve is closed, the force system has a single resultant, which passes through P.

CHAPTER XIV.

THE PRINCIPLE OF VIRTUAL WORK APPLIED TO ANY SYSTEM OF BODIES.

259.] **Forces applied to a Particle.** It has been shown in Art. 199 that the resultant of any number of forces applied to a particle may be represented by the side required to close the polygon of the forces. And whether the polygon $OP_1 P_2 \dots P_n$ be plane or gauche, it is clear (as in Art. 55) that the sum of the projections of the sides, taken in order, along any line OA, is equal to zero.

Let the projections of the sides be denoted by $Q_1, Q_2, \dots Q_n$. Then $Q_1 + Q_2 + \dots + Q_n = 0$. Multiplying this by OA, an arbitrary length along the line OA, we have

$$Q_1 \,.\, OA + Q_2 \,.\, OA + \dots + Q_n \,.\, OA = 0.$$

But if p_1 is the projection of OA along OP_1, we have (see Art. 56)

$$Q_1 \,.\, OA = OP_1 \,.\, p_1.$$

If, then, the sides OP_1, $P_1P_2, \dots$ be denoted by P_1, $P_2, \dots$ we have

$$P_1 \,.\, p_1 + P_2 \,.\, p_2 + \dots + P_n \,.\, p_n = 0;$$

and if the sides represent forces, each term in this equation is the virtual work of the corresponding force for the displacement OA. Since the resultant, R, of $n-1$ of the forces is $-P_n$, we have

$$R \,.\, r = P_1 \,.\, p_1 + P_2 \,.\, p_2 \dots;$$

and if the displacement is small, this equation is written (as in Art. 64)

$$R\delta r = P_1 \delta p_1 + P_1 \delta p_2 + \dots. \tag{1}$$

In particular, if X, Y, Z denote the rectangular components of R, we have

$$R\delta r = X\delta x + Y\delta y + Z\delta z. \tag{2}$$

260.] **Extension to any number of Connected Particles.** If two particles, m_1 and m_2, are connected by a rigid inextensible rod, and are in equilibrium under the action of forces, P_1, $Q_1, \dots$

applied to m_1 and $P_2, Q_2, \ldots$ applied to m_2, it is evident (as in Art. 105) that the force arising from the connexion acts in the line joining m_1 to m_2. If, then, this force be denoted by T, and the distance between the particles by r, we have for the equilibrium of m_1
$$P_1 \delta p_1 + Q_1 \delta q_1 + \ldots + T\delta_1 r = 0,$$
$\delta_1 r$ denoting the change in r arising from an arbitrary small displacement of m_1. The equation of equilibrium of m_2 is
$$P_2 \delta p_2 + Q_2 \delta q_2 + \ldots + T\delta_2 r = 0;$$
and if in the new positions of m_1 and m_2 the distance between them remains unaltered, $\delta_1 r + \delta_2 r = 0$. Hence, by addition, from these equations we obtain the equation
$$P_1 \delta p_1 + Q_1 \delta q_1 + \ldots + P_2 \delta p_2 + Q_2 \delta q_2 + \ldots = 0, \qquad (1)$$
which is free from the internal force T.

This is exactly the same as the investigation already given for coplanar forces in Chap. VI. The extension to any number of particles, that is, to any extended body, proceeds just as in that chapter, and the enunciation of the principle of virtual work there given applies in general without the limitation that the forces are coplanar.

If in the case of the two particles m_1 and m_2, considered above, their new positions are such that the distance between them is altered by δr, the equation of virtual work will be
$$P_1 \delta p_1 + Q_1 \delta q_1 + \ldots + P_2 \delta p_2 + Q_2 \delta q_2 + \ldots + T\delta r = 0; \qquad (2)$$
and, generally, if the virtual displacement is such that the internal forces do virtual work, these forces will enter into the equation of virtual work in exactly the same manner as the applied forces. The theorem of virtual work may, therefore, be thus enunciated:—

When a material system is in equilibrium under the action of any external and internal forces, the sum of the virtual works of the external and internal forces is equal to zero for any small virtual displacement whatsoever.

Instead of saying that the total virtual work is *zero*, we should in strictness say that it is an indefinitely small quantity of the second order, the greatest of the displacements being considered as a small quantity of the first order. This has been already explained in vol. i.

The proof of the converse proposition—namely, that when the virtual work vanishes for all imagined displacements, the system

is in equilibrium—has been already given in Art. 108 for coplanar forces; and as the proof obviously holds for non-coplanar forces, it is unnecessary to reproduce it here.

261.] **Displacements along Smooth Surfaces.** If any body or system of connected bodies be in contact with smooth curves or surfaces, and the system be imagined to receive any small displacement along these curves or surfaces, it is clear that, since the point of application of each of the geometrical forces (reactions of the curves or surfaces) moves in a plane at right angles to the corresponding force, these forces will contribute nothing to the equation of virtual work for such a displacement.

If any of the bodies of the system are connected by strings or rods whose lengths are unaltered in the virtual displacement chosen, the tensions of these strings or rods will not enter into the equation of virtual work. But, as already explained in Arts. 73 and 107, we may choose virtual displacements of the system which violate the imposed conditions at the expense of bringing into our equation the corresponding forces.

262.] **Kinematical Theorem I.** When all the points of a rigid body move parallel to a plane, the motion may be produced by a pure rotation round an axis perpendicular to this plane.

DEF. A motion of a body round an axis whereby each point in the body describes an arc of a circle having its centre on the axis and its plane perpendicular to it is called *pure rotation*.

The position of the body will evidently be known if the positions of any two points in a plane parallel to the plane of motion are known.

Let A and B be any two points in such a plane, and suppose that after the displacement of the body they occupy the positions A' and B' (Fig. 252). At the middle points of AA' and BB' erect two perpendiculars, which meet in I. Then in the triangles AIB and $A'IB'$, $AI = A'I$, $BI = B'I$, and $AB = A'B'$; therefore the triangle $A'IB'$ is nothing more than AIB turned round the point I through an angle AIA' or BIB'. Hence the line AB can be brought into its new position by a pure rotation about I, and the same is true of every point rigidly connected with A and B in the plane AIB.

If through I an axis be drawn perpendicular to the plane of motion, it is evident that the body can be brought into its new position by a pure rotation about this axis through an angle

$= AIA'$, however complicated the paths along which A and B have travelled to A' and B'.

When the motion of the body is small, this axis is called the *Instantaneous Axis*; and it is obviously constructed by *drawing two planes normal to the lines of motion of any two points in the body.* The intersection of these planes is the instantaneous axis.

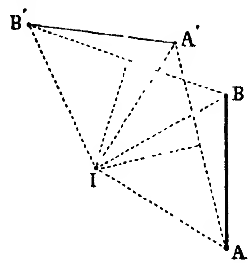

Fig. 252.

When the body is a plane figure, the point I is called the *Instantaneous Centre*; and the consideration of this point is of very extensive use in Kinematics, Statics, and Geometry.

To construct the instantaneous centre, *at any two points erect perpendiculars to the lines of motion of these points, and their intersection is the required point.*

263.] **Kinematical Theorem II.** The motion of a rigid body round a fixed point is at every instant a pure rotation round an axis.

One point, O, in the body being fixed, the position of the body will be known if the positions of any two points, A and B, not in directum with O are known.

Round O let a sphere, forming part of the body or rigidly connected with it, be described with arbitrary radius, and let A and B (Fig. 252) be any two points on the sphere. After the motion of the body let A' and B' be the positions of A and B. Imagine the lines AB, $A'B'$, AA', and BB' in this figure to be arcs of great circles on the sphere instead of right lines. Then, at the middle points of AA' and BB' draw two great circles perpendicular to AA' and BB', respectively, and let them meet in I. In exactly the same way as in the last theorem, we have the spherical triangles AIB and $A'IB'$ equal; that is, the latter triangle is the former turned round the axis OI through an angle AIA' or BIB'. Hence the whole body is brought by rotation through this angle round the axis OI from the old to the new position.

264.] **Kinematical Theorem III.** If a body has a motion of translation represented in magnitude and direction by a right line OA, and at the same time a motion of translation represented in magnitude and direction by a right line OB, the

resulting motion of translation is represented in magnitude and direction by the diagonal, OC, of the parallelogram determined by OA and OB.

This proposition has been already illustrated in Art. 11. It follows immediately that any motion of translation can be resolved by the parallelepiped law into three motions along the axes of x, y, and z, after the manner of forces.

265.] **Kinematical Theorem IV.** If a body receives a motion of rotation round an axis OA, the rotation being represented in magnitude by OA,—i. e. so many units of circular measure being represented by so many centimetres, the scale being, of course, quite arbitrary—and at the same time a motion of rotation (of the same sign as the first) round an axis OB, the rotation being represented in magnitude by OB, the resulting motion is one of rotation round the diagonal, OC, of the parallelogram determined by OA and OB, and is represented in magnitude by this diagonal.

[The signs of rotations are determined by the rule given in Art. 200. We shall, for definiteness, suppose that when a watch is held with its face perpendicular to AO, so that OA passes up through the glass, the rotation about OA takes place in a sense opposite to that of the hands; and similarly for OB.]

Let P be any point on OC, p the perpendicular from P on OA, q the perpendicular from P on OB, and $k \cdot OA$ and $k \cdot OB$ the angular motions round OA and OB, respectively. Then in virtue of the rotation round OA, P moves upwards from the plane of the paper through a distance equal to $kp \cdot OA$; and in virtue of the rotation round OB, P moves downwards from the plane of the paper through a distance equal to $kq \cdot OB$. Therefore the whole motion of P upwards is equal to

$$k(p \cdot OA - q \cdot OB).$$

But this is obviously zero; therefore P is at rest, and so is every point on OC. The motion is, then, a rotation round OC. Let Ω be the angular rotation of the body round OC. Then the point A moves upwards from the plane of the paper through a distance equal to $\Omega \cdot OA \sin AOC$, since $OA \sin AOC =$ the perpendicular from A on OC. But A in turning round OB moves through a distance equal to $k \cdot OB \cdot OA \times \sin AOB$. Hence

$$\Omega \cdot OA \sin AOC = k \cdot OB \cdot OA \sin AOB,$$

or
$$\Omega = k \,.\, OB \,.\, \frac{\sin AOB}{\sin AOC}$$
$$= k \,.\, OC.$$

Therefore the resulting angular velocity is represented by OC, if the component rotations are represented by OA and OB.

This proposition is known as the 'parallelogram of angular velocities.' It follows at once that an angular motion about any axis, OL, may be decomposed into three angular motions about three axes, Ox, Oy, and Oz. If these latter are rectangular, an angular motion ω about OL is equivalent to angular motions, $\omega \cos \alpha$, $\omega \cos \beta$, and $\omega \cos \gamma$, of the same sign, round the axes of x, y, and z, the direction angles of OL being α, β, γ.

266.] **General Displacement of a Rigid Body.** The position of every point in a rigid body is known when the positions of any three points in it are known, provided that these points are not in one right line. The general displacement of a rigid body is, therefore, the same as that of a system of three points forming a triangle.

Let A, B, C be the positions of three points in the body before the displacement, and A', B', C' the positions occupied by these points after the displacement. Then the triangle ABC may be brought into the position $A'B'C'$ by moving A directly to A' while B and C move parallel to AA' through distances equal to AA', and then turning the triangle about A' until B and C coincide with B' and C'. But (Art. 263) this latter motion is one of rotation round some axis through A'. Hence *the general displacement of a rigid body consists of a motion of translation which is the same for all its points, and a motion of rotation round an axis through an angle which is the same for all its points.*

To find the changes produced in the co-ordinates, x, y, z, of any point in the body by a general displacement, we may consider the motions of translation and of rotation separately.

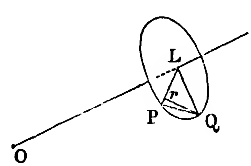

Fig. 253.

Although we shall be concerned only with small displacements, it is well to investigate the changes produced in the co-ordinates of a point by a rotation through any angle, θ, round an axis whose position is given.

Let the direction angles of the axis, OL (Fig. 253), be α, β, γ;

let P be the point (x, y, z) which, after the body has rotated through an angle θ round OL, occupies the position Q; let PL $(=p)$ be the perpendicular from P on OL, and Qr a perpendicular from Q on LP. Now the x of Q is the projection of OQ on the axis of x; therefore the change in x is the projection of PQ along Ox, or the sum of the projections of Pr and rQ. But $Pr = p\,(1-\cos\theta)$, and $Qr = p\sin\theta$.

Again, if the direction angles of PL are λ, μ, ν, since Qr is at right angles to OL and PL, the direction cosines of Qr are $\cos\beta\cos\nu - \cos\gamma\cos\mu$, &c. Hence, if the x of Q is x',

$$x' - x = p\sin\theta\,(\cos\beta\cos\nu - \cos\gamma\cos\mu) - 2p\cos\lambda\sin^2\tfrac{1}{2}\theta. \quad (1)$$

But $p\cos\lambda$ is the projection of PL along the axis of x, or the projection of OP — the projection of OL, and since $OL = x\cos\alpha + y\cos\beta + z\cos\gamma$,

$$p\cos\lambda = x - (x\cos\alpha + y\cos\beta + z\cos\gamma)\cos\alpha.$$

Similarly

$$p\cos\mu = y - (x\cos\alpha + y\cos\beta + z\cos\gamma)\cos\beta,$$

$$p\cos\nu = z - (x\cos\alpha + y\cos\beta + z\cos\gamma)\cos\gamma.$$

Substituting these values in (1), we have

$$x' - x = \sin\theta\,(z\cos\beta - y\cos\gamma) + 2\sin^2\tfrac{1}{2}\theta\left[(x\cos\alpha + y\cos\beta + z\cos\gamma)\cos\alpha - x\right], \quad (2)$$

and similar values for the changes in y and z.

If the angular rotation θ is very small, we have

$$\delta x = (z\cos\beta - y\cos\gamma)\,\delta\theta,$$

$$\delta y = (x\cos\gamma - z\cos\alpha)\,\delta\theta,$$

$$\delta z = (y\cos\alpha - x\cos\beta)\,\delta\theta,$$

and if the components of the rotation $\delta\theta$ along the axes be denoted by $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$, these equations give

$$\left.\begin{aligned} \delta x &= z\delta\theta_2 - y\delta\theta_3 \\ \delta y &= x\delta\theta_3 - z\delta\theta_1 \\ \delta z &= y\delta\theta_1 - x\delta\theta_2 \end{aligned}\right\}. \quad (3)$$

Of course these equations can be obtained very simply by considering the separate changes in the co-ordinates produced by successive rotations $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$ round the axes of x, y, z respectively. (See Routh's *Rigid Dynamics*.)

If the components of the motion of translation common to all points in the body be δa, δb, δc, the complete changes in the co-ordinates for a small displacement will be

$$\left.\begin{aligned}\delta x &= \delta a + z\delta\theta_2 - y\delta\theta_3\\ \delta y &= \delta b + x\delta\theta_3 - z\delta\theta_1\\ \delta z &= \delta c + y\delta\theta_1 - x\delta\theta_2\end{aligned}\right\} \qquad (4)$$

267.] **Deduction of the Six Equations of Equilibrium.** Replacing the virtual work of each force in equation (1) of Art. 260 by the virtual work of its three components, the general equation of virtual work becomes

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = 0, \qquad (1)$$

and substituting in this equation the values of δx, δy, and δz given by (4), we have

$$\begin{aligned}\delta a\,.\,\Sigma X + \delta b\,.\,\Sigma Y + \delta c\,.\,\Sigma Z + \delta\theta_1\,.\,\Sigma\,(Zy - Yz)&\\ + \delta\theta_2\,.\,\Sigma(Xz - Zx) + \delta\theta_3\,.\,\Sigma(Yx - Xy) &= 0. \qquad (2)\end{aligned}$$

Now, the displacement being quite arbitrary, its components δa, δb, δc, $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$, are completely independent. Hence in (2) we may consider all of them zero except one, and the equation then gives the coefficient of this one equal to zero. Thus (2) involves the six equations

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

$$\Sigma(Zy - Yz) = 0, \; \Sigma(Xz - Zx) = 0, \; \Sigma(Yx - Xy) = 0,$$

which are the equations of equilibrium before obtained (see Art. 240).

268.] **Method of Lagrange.** Lagrange based the whole of Dynamics—alike its applications to the equilibrium and motion of rigid bodies, of inextensible and extensible strings, of elastic rods and membranes, of fluids, and of elastic media propagating disturbances by undulatory motions—on the single Principle of Virtual Work. So far as the equilibrium problem is concerned, in its reference to any of the material systems just named, the idea of the method is shortly this—

Imagine the system to have taken its position or configuration of equilibrium; then imagine any small derangement whatever of the points, or infinitesimal elements, of the system; calculate the total quantity of work, both of the external forces applied to the system

and of its internal forces (forces mutually exerted by neighbouring parts of the system), and equate to zero this sum total of work.

Now the system whose equilibrium is proposed for investigation in any case may be one in which certain specified geometrical conditions have to be satisfied—as, for instance, a system of particles connected by inextensible flexible strings or inextensible and inflexible rods—and, as has been abundantly illustrated in the earlier parts of this work, we may either respect the imposed geometrical conditions (as it is often convenient to do when we merely seek for *positions* of equilibrium), or we may imagine a derangement of the parts of the system in which no regard is paid to these imposed conditions. But if we do the latter, it is at the expense of introducing into our equation of Virtual Work the work which would be done by an internal force whose existence is a necessary consequence of the particular geometrical condition under consideration. The imposition of every geometrical condition in a system establishes the existence of an internal force in the system; and the examples hitherto treated have related to the simpler cases in which such forces are due to the invariability of distances between particles or the restriction of the positions of particles to smooth surfaces.

We now proceed to consider, after the manner of Lagrange, the theory of all imposed geometrical conditions for a system of particles in a general manner.

269.] **Equations of Condition may be replaced by Forces.** Suppose a system of n particles whose co-ordinates are connected by k equations of condition,

$$L_1 = 0, \quad L_2 = 0, \dots L_k = 0, \tag{1}$$

each of these equations being of the form

$$f(x_1, y_1, z_1, x_2, y_2, z_2, \dots x_n, y_n, z_n) = 0,$$

that is, involving the co-ordinates of all the points in general. Then the equation of virtual work for the position of equilibrium of the system is

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = 0,$$

which, when written at full length, is

$$X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1 + \dots + X_n\delta x_n + Y_n\delta y_n + Z_n\delta z_n = 0. \tag{2}$$

Now *if the virtual displacements of the particles were all independent*, this equation would involve the vanishing of the coefficient of each displacement (see Art. 267); but the displacements of the particles must be such as still to satisfy the equations (1). Hence the quantities δx, &c., are connected by the k equations

$$\left.\begin{array}{l}
\dfrac{dL_1}{dx_1}\delta x_1 + \dfrac{dL_1}{dy_1}\delta y_1 + \dfrac{dL_1}{dz_1}\delta z_1 + \dots \\[2ex]
+\dfrac{dL_1}{dx_n}\delta x_n + \dfrac{dL_1}{dy_n}\delta y_n + \dfrac{dL_1}{dz_n}\delta z_n = 0, \\[2ex]
\dfrac{dL_2}{dx_1}\delta x_1 + \dfrac{dL_2}{dy_1}\delta y_1 + \dfrac{dL_2}{dz}\delta z_1 + \dots \\[2ex]
+\dfrac{dL_2}{dx_n}dx_n + \dfrac{dL_2}{dy_n}\delta y_n + \dfrac{dL_2}{dz_n}\delta z_n = 0, \\[2ex]
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\[2ex]
\dfrac{dL_k}{dx_1}\delta x_1 + \dfrac{dL_k}{dy_1}\delta y_1 + \dfrac{dL_k}{dz_1}\delta z_1 + \dots \\[2ex]
+\dfrac{dL_k}{dx_n}\delta x_n + \dfrac{dL_k}{dy_n}\delta y_n + \dfrac{dL_k}{dz_n}\delta z_n = 0.
\end{array}\right\} \qquad (3)$$

Solving these k equations for any k of the displacements—suppose $\delta x_1, \delta x_2, \dots \delta x_k$—and substituting their values in (2), we obtain an equation connecting the remaining $3n-k$ displacements of the form

$$\begin{aligned}
A_{k+1}\delta x_{k+1} &+ \dots + A_n \delta x_n \\
&+ B_1 \delta y_1 + \dots + B_n \delta y_n \\
&+ C_1 \delta z_1 + \dots + C_n \delta z_n = 0.
\end{aligned} \qquad (4)$$

Now, the remaining quantities, δx_{k+1}, &c., are completely independent, and therefore (see Art. 267) every coefficient in this equation must $= 0$. Thus, we obtain $3n-k$ equations involving the forces, that is, statical equations of condition. Combining these statical equations with the equations of connexion (1), we have finally $3n$ equations for the $3n$ co-ordinates of the particles. The elimination of the displacements from the equations can, however, be exhibited in a more symmetrical and useful form.

Multiply the left sides of the equations (3) by $\lambda_1, \lambda_2, \dots \lambda_k$ in

order, these multipliers being undetermined quantities; then add all the results together, and finally equate to zero the coefficient of every displacement in the resulting equation. Thus we shall have the following $3n$ equations:—

$$\left.\begin{array}{l} X_1 + \lambda_1 \dfrac{dL_1}{dx_1} + \lambda_2 \dfrac{dL_2}{dx_1} + \ldots + \lambda_k \dfrac{dL_k}{dx_1} = 0, \\ Y_1 + \lambda_1 \dfrac{dL_1}{dy_1} + \lambda_2 \dfrac{dL_2}{dy_1} + \ldots + \lambda_k \dfrac{dL_k}{dy_1} = 0, \\ Z_1 + \lambda_1 \dfrac{dL_1}{dz_1} + \lambda_2 \dfrac{dL_2}{dz_1} + \ldots + \lambda_k \dfrac{dL_k}{dz_1} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \qquad (5)$$

If from these equations we eliminate the k undetermined multipliers, we shall have $3n - k$ statical equations of condition, as before; and we see, as an essential characteristic of the Lagrangian method, that *the variations of the co-ordinates of all the particles of the system may be treated as independent quantities after we have taken the variations of the left-hand sides of all the equations of condition, multiplied each of these by an arbitrary multiplier, and included the products in one equation of Virtual Work.*

Now, this method of elimination has the advantage of discovering the geometrical forces, or forces arising from the connexions, of the system. For, suppose that we suppress the condition $L_1 = 0$; then the system will begin to move; but it may be kept at rest by applying a special force to each particle.

Let the components of the force applied to m_1 be X_1', Y_1', Z_1', those of the force applied to m_2, X_2', Y_2', Z_2', and so on for all the others. The equations of equilibrium of m_1 will then be

$$X_1 + X_1' + \lambda_2 \frac{dL_2}{dx_1} + \ldots + \lambda_k \frac{dL_k}{dx_1} = 0,$$

$$Y_1 + Y_1' + \lambda_2 \frac{dL_2}{dy_1} + \ldots + \lambda_k \frac{dL_k}{dy_1} = 0,$$

$$Z_1 + Z_1' + \lambda_2 \frac{dL_2}{dz_1} + \ldots + \lambda_k \frac{dL_k}{dz_1} = 0,$$

similar equations holding for the other particles.

Subtracting the left side of each of these from that of the corresponding equation in (5), we have

$$X_1' = \lambda_1 \frac{dL_1}{dx_1},$$

$$Y_1' = \lambda_1 \frac{dL_1}{dy_1},$$

$$Z_1' = \lambda_1 \frac{dL_1}{dz_1}.$$

Hence $$X_1' : Y_1' : Z_1' = \frac{dL_1}{dx_1} : \frac{dL_1}{dy_1} : \frac{dL_1}{dz_1},$$

and $$\sqrt{X_1'^2 + Y_1'^2 + Z_1'^2} = \lambda_1 \sqrt{\left(\frac{dL_1}{dx_1}\right)^2 + \left(\frac{dL_1}{dy_1}\right)^2 + \left(\frac{dL_1}{dz_1}\right)^2}.$$

If, now, all the co-ordinates involved in the equation $L_1 = 0$ are considered constant except x_1, y_1, and z_1, this equation will denote a surface on which the particle m_1 is constrained to lie, and

$$\frac{dL_1}{dx_1}, \quad \frac{dL_1}{dy_1}, \quad \frac{dL_1}{dz_1},$$

each divided by $$\sqrt{\left(\frac{dL_1}{dx_1}\right)^2 + \left(\frac{dL_1}{dy_1}\right)^2 + \left(\frac{dL_1}{dz_1}\right)^2},$$

will be the direction-cosines of the normal to this surface at the point (x_1, y_1, z_1). It is evident, therefore, that the force required to keep the particle m_1 at rest, when the condition $L_1 = 0$ is suppressed, is a force acting normally to this surface, its magnitude being

$$\lambda_1 \sqrt{\left(\frac{dL_1}{dx_1}\right)^2 + \left(\frac{dL_1}{dy_1}\right)^2 + \left(\frac{dL_1}{dz_1}\right)^2}.$$

In the same way the force required to keep m_2 at rest acts normally to the surface denoted by $L_1 = 0$ when x_2, y_2, z_2 are considered as the only variable co-ordinates in the equation, and the magnitude of this force is

$$\lambda_1 \sqrt{\left(\frac{dL_1}{dx_2}\right)^2 + \left(\frac{dL_1}{dy_2}\right)^2 + \left(\frac{dL_1}{dz_2}\right)^2}.$$

If the condition $L_2 = 0$ were suppressed, it follows in like manner that forces

$$\lambda_2 \sqrt{\left(\frac{dL_2}{dx_1}\right)^2 + \left(\frac{dL_2}{dy_1}\right)^2 + \left(\frac{dL_2}{dz_1}\right)^2}, \text{ \&c.},$$

should be applied to the particles m_1, &c., in directions normal to the surfaces represented by the equation $L_2 = 0$ when the sole variables in it are the co-ordinates of m_1, &c., in succession. It is easy to see that

$$\lambda_1 \left(\frac{dL_1}{dx_1}\delta x_1 + \frac{dL_1}{dy_1}\delta y_1 + \frac{dL_1}{dz_1}\delta z_1\right)$$

is equal to $\quad F_1 (\cos\alpha \,.\, \delta x_1 + \cos\beta \,.\, \delta y_1 + \cos\gamma \,.\, \delta z_1)$,

where F_1 is the force of connexion acting on m_1 in virtue of the condition $L_1 = 0$, and α, β, γ the direction angles of the normal to the surface denoted by $L_1 = 0$ when the co-ordinates of m_1 are regarded as the only variables in it.

Now, the multiplier of F_1 in this expression is evidently the projection of the displacement of m_1 along the normal to this surface. If this projection be denoted by δn, n being the length of the normal at the position of m_1 measured from some fixed point on the normal, we have

$$\lambda_1 \delta L_1 = F_1 \delta n,$$

in which the variation of L_1 has reference solely to the particle m_1.

The right-hand side of this equation at once identifies the term $\lambda_1 \delta L_1$ with the virtual work of an internal force, since $F_1 \delta n$ is explicitly such; and this force acts along the direction in which the function L_1 varies most rapidly (i.e. the normal to the surface denoted by the equation $L_1 = 0$).

Hence Lagrange habitually speaks of such a term as $\lambda\delta L$ in the equation of virtual work as 'the virtual moment of a force tending to vary the function L.'

Examples.

1. A number of heavy particles are attached at given intervals to a weightless string the extremities of which are fixed; investigate the circumstances of equilibrium (Funicular Polygon).

Let (a, b) be the co-ordinates of one of the fixed extremities, (x_1, y_1), (x_2, y_2), ... the co-ordinates of the particles taken in order from this extremity, l_{01}, l_{12}, ... the lengths of the portions of the string between these points, and W_1, W_2, ... the weights of the particles.

Then the equations of connexion of the system are

$$(a-x_1)^2 + (b-y_1)^2 = l_{01}^2,$$
$$(x_1-x_2)^2 + (y_1-y_2)^2 = l_{12}^2, \text{ \&c.}$$

Hence the Lagrangian equation of virtual work is

$$W_1\delta y_1 + W_2\delta y_2 + \ldots - \lambda_1 \{(a-x_1)\delta x_1 + (b-y_1)\delta y_1\}$$
$$+\lambda_2\{(x_1-x_2)(\delta x_1 - \delta x_2) + (y_1-y_2)(\delta y_1 - \delta y_2)\} + \ldots = 0.$$

Equate to zero the coefficients of the several displacements: then

$$\lambda_1(a-x_1) - \lambda_2(x_1-x_2) = 0,$$
$$\lambda_2(x_1-x_2) - \lambda_3(x_2-x_3) = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$W_1 - \lambda_1(b-y_1) + \lambda_2(y_1-y_2) = 0,$$
$$W_2 - \lambda_2(y_1-y_2) + \lambda_3(y_2-y_3) = 0.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The first set of these equations evidently gives

$$\lambda_1(a-x_1) = \lambda_2(x_1-x_2) = \lambda_3(x_2-x_3) = \ldots = T, \text{ suppose,}$$

and by substituting in the remaining set,

$$\frac{b-y_1}{a-x_1} = \frac{y_1-y_2}{x_1-x_2} + \frac{W_1}{T},$$
$$\frac{y_1-y_2}{x_1-x_2} = \frac{y_2-y_3}{x_2-x_3} + \frac{W_2}{T}.$$

But $\dfrac{b-y_1}{a-x_1}$ is the tangent of the inclination of the portion l_{01} of the string to the horizon. Hence we have

$$\tan\theta_{01} = \tan\theta_{12} + \frac{W_1}{T},$$
$$\tan\theta_{12} = \tan\theta_{23} + \frac{W_2}{T},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

as in Art. 35. Also the tension of the string joining (a, b) to (x_1, y_1) is $\dfrac{\lambda_1}{l_{01}}$ acting from the first point towards the second, and so on for the other tensions.

2. Deduce by the method of Lagrange the conditions of equilibrium of a system of three particles forming a rigid triangle, each particle being acted on by given forces.

Let (x_1, y_1, z_1) be the co-ordinates of one particle, and (X_1, Y_1, Z_1) the components of the force acting on it, with similar notation for the other two particles. Then, if l_{12}, l_{23}, l_{31} denote the sides of the triangle, the equations of connexion are

$$(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 = l_{12}{}^2,$$
$$(x_2-x_3)^2 + (y_2-y_3)^2 + (z_2-z_3)^2 = l_{23}{}^2,$$
$$(x_3-x_1)^2 + (y_3-y_1)^2 + (z_3-z_1)^2 = l_{31}{}^2.$$

Hence the Lagrangian equation of equilibrium is

$$X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1 + \ldots + \lambda_{12}\{(x_1-x_2)(\delta x_1 - \delta x_2)$$
$$+ (y_1-y_2)(\delta y_1 - \delta y_2) + (z_1-z_2)(\delta z_1 - \delta z_2)\} + \ldots = 0,$$

the undetermined multipliers being λ_{12}, λ_{23}, and λ_{31}.

Equating to zero the coefficients of the displacements, we have

$$X_1+\lambda_{12}(x_1-x_2)-\lambda_{31}(x_3-x_1)=0, \tag{1}$$

$$Y_1+\lambda_{12}(y_1-y_2)-\lambda_{31}(y_3-y_1)=0, \tag{2}$$

$$Z_1+\lambda_{12}(z_1-z_2)-\lambda_{31}(z_3-z_1)=0, \tag{3}$$

with similar equations for the other particles.

By addition, we have at once

$$X_1+X_2+X_3=0, \text{ or } \Sigma X=0,$$

$$Y_1+Y_2+Y_3=0, \text{ or } \Sigma Y=0,$$

$$Z_1+Z_2+Z_3=0, \text{ or } \Sigma Z=0,$$

which are the ordinary equations of translation.

Again, multiplying (1) by y_1 and (2) by x_1, and subtracting,

$$Y_1x_1-X_1y_1-\lambda_{12}(x_1y_2-y_1x_2)-\lambda_{31}(x_1y_3-y_1x_3)=0,$$

and by taking the similar equations for the other particles, and adding, we get $\qquad \Sigma(Yx-Xy)=0.$

Similarly, $\qquad \Sigma(Xz-Zx)=0,$

and $\qquad \Sigma(Zy-Yz)=0.$

These last three are the equations of moments, and they constitute, with the first three, six equations of equilibrium. Now these are all the conditions that can be obtained among the forces and co-ordinates. For if n particles be connected by k equations of condition, there are (Art. 269), $3n-k$ final equations. But here $n=3$, $k=3$, therefore $3n-k=6$. It is to be observed that the equations of equilibrium of any rigid body must be the same in number as those for three particles forming a rigid triangle, because if three points of a rigid body are determined in position, the position of the body is determined.

3. Show that the equations of equilibrium of a system subject to given conditions may be expressed as the vanishing of the differential coefficients of a single function of the co-ordinates of the system.

Suppose that

$$(X_1dx_1+Y_1dy_1+Z_1dz_1)+(X_2dx_2+Y_2dy_2+Z_2dz_2)+\ldots,$$

or $\Sigma(Xdx+Ydy+Zdz), \equiv -d\Pi$ where Π is a function of the co-ordinates $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ Then taking

$$U\equiv -\Pi+\lambda_1L_1+\lambda_2L_2+\ldots,$$

where $L_1=0$, $L_2=0$, ... are the equations of condition, we shall have

$$\frac{dU}{dx_1}\equiv X_1+\lambda_1\frac{dL_1}{dx_1}+\lambda_2\frac{dL_2}{dx_1}+\ldots+L_1\frac{d\lambda_1}{dx_1}+L_2\frac{d\lambda_2}{dx_1}+\ldots.$$

But since the co-ordinates make $L_1=L_2=\ldots=0$,

$$\frac{dU}{dx_1}\equiv X_1+\lambda_1\frac{dL_1}{dx_1}+\lambda_2\frac{dL_2}{dx_1}+\ldots,$$

and comparing with equations (5), we see that the equations of equilibrium are

$$\frac{dU}{dx_1}=0, \frac{dU}{dx_2}=0, \ldots \frac{dU}{dy_1}=0, \frac{dU}{dy_2}=0, \text{ \&c.}$$

270.] Distinctive Feature of the Lagrangian Method. If the first method of eliminating the displacements described in the last Article is adopted, we arrive at an equation such as (4) of that Article, from which the conditions of equilibrium are obtained by equating to zero the coefficients of the displacements. But in proceeding thus, we fail to obtain the values of the internal and geometrical forces of the system. Now these forces are, as we have seen, intimately related to the undetermined multipliers; and as these latter are found from the Lagrangian equations, it follows that—

The method of Lagrange gives not only the conditions of equilibrium, but also the internal forces of the system.

A single very elementary example will suffice to render this clear.

Two heavy particles of weights W_1 and W_2 are connected by a rigid rod, and each particle rests on a smooth inclined plane. The inclinations of the planes are i_1 and i_2 and their intersection is horizontal; find the position of equilibrium and the internal and geometrical forces.

Let the line of intersection of the planes be taken as axis of z, let the axis of y be vertical and that of x horizontal. Also let $(x_1\ y_1\ z_1)$ $(x_2\ y_2\ z_2)$ be the co-ordinates of the particles, and l the length of the rod connecting them. Then the equations of connexion are

$$y_1 - x_1 \tan i_1 = 0,$$

$$y_2 + x_2 \tan i_2 = 0,$$

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2 = l^2.$$

Hence the Lagrangian equation of equilibrium is

$$-W_1\delta y_1 - W_2\delta y_2 + \lambda_1(\delta y_1 - \tan i_1 . \delta x_1) + \lambda_2(\delta y_2 + \tan i_2 . \delta x_2)$$
$$+\tau\{(x_1-x_2)(\delta x_1-\delta x_2)+(y_1-y_2)(\delta y_1-\delta y_2)+(z_1-z_2)(\delta z_1-\delta z_2)\} = 0,$$

λ_1, λ_2, and τ being the undetermined multipliers.

Equating to zero the coefficients of the separate displacements,

$$-W_1+\lambda_1+\tau(y_1-y_2) = 0,$$

$$-W_2+\lambda_2-\tau(y_1-y_2) = 0,$$

$$\lambda_1 \tan i_1 - \tau(x_1-x_2) = 0,$$

$$\lambda_2 \tan i_2 - \tau(x_1-x_2) = 0,$$

$$\tau(z_1-z_2) = 0.$$

From the last equation we have $z_1-z_2 = 0$, which shows that both particles must lie in a vertical plane perpendicular to the line of intersection of the inclined planes.

If θ be the inclination of the line joining the particles to the horizon, the other equations give

$$(W_1+W_2)\tan\theta = W_1\cot i_2 - W_2\cot i_1,$$

$$\tau l = \frac{W_1\sin i_1}{\cos(i_1-\theta)},$$

$$\lambda_1 = \frac{W_1\cos\theta\cos i_1}{\cos(i_1-\theta)},$$

$$\lambda_2 = \frac{W_2\cos\theta\cos i_2}{\cos(i+\theta)}.$$

The reader will easily perceive that τl is the tension of the rod, and $\lambda_1\sec i_1$ and $\lambda_2\sec i_2$ the reactions of the smooth planes. Thus we have the same values of the inclination of the rod and of the internal forces as we should have obtained by the ordinary statical methods.

Now suppose that the equation of virtual work is employed according to the first method; that is, let us write

$$W_1\delta y_1 + W_2\delta y_2 = 0,$$

$$\delta y_1 - \tan i_1 . \delta x_1 = 0,$$

$$\delta y_2 + \tan i_2 . \delta x_2 = 0,$$

$$(x_1-x_2)(\delta x_1-\delta x_2)+(y_1-y_2)(\delta y_1-\delta y_2)+(z_1-z_2)(\delta z_1-\delta z_2) = 0,$$

and eliminate the displacements without employing undetermined multipliers. Then we obtain simply the equations

$$z_1 - z_2 = 0,$$

$$(W_1+W_2)\tan\theta = W_1\cot i_2 - W_2\cot i_1,$$

which define the position of equilibrium, without giving the values of the unknown forces of the system.

271.] **Work.** If a force, R, acts at a point (x, y, z) which, from any cause, receives a small displacement whose projections on the axes of co-ordinates are dx, dy, dz, and if the components of R are X, Y, Z, the work actually done by the force is

$$Xdx + Ydy + Zdz. \tag{1}$$

If a force P which is constant both in magnitude and line of action acts at a point, A, which from any cause is displaced through any distance, AB, along the line of action and in the sense of P, the whole amount of work done by the force is

$$P \times AB;$$

and if the displacement takes place in the sense opposite to that of P, the work done by P is $-P\times AB$.

If the force P is constant in magnitude and direction (but not line of action) while its point, A, of application is displaced along

any curve, AB (Fig. 254), the work done by the force (which is the integral of all the elements of work done during the passage) is

$$P \times \text{projection of } AB \text{ along the direction of } P.$$

As an instance, take the case of a heavy body of weight W whose centre of gravity occupies the point A initially. If the body is displaced along any curve or surface whatever, so that its centre of gravity finally occupies the position B, the work done by W is

$$W \times h,$$

where h is the excess of the height of A over that of B; so that W does positive work if B is below A, negative work if B is above A, and no work if A and B are at the same horizontal level. Similarly in Fig. 254, the working force being constant in magnitude and direction, if AD is perpendicular to P, no work is done on the whole in the passage from A to D.

If the working force, P, is constant in magnitude and variable in direction, while its point of application is at each instant moving along the line of action of P, the work done by P from one point A to another B is the product $P.s$, where s is the whole length of the path of the point of application between A and B. For instance, a constant pressure, P, exerted on the arm of a capstan.

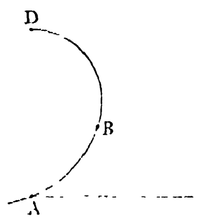

Fig. 254.

If the working force varies both in magnitude and in direction while its point of application describes any path between a point A and a point B, the total work must be obtained by taking the elementary work done by the force for a very small displacement of its point of application, and integrating this. We may at each point resolve the force into three components, so that the element of work is expressed by (1), and the total work done between A and B is

$$\int_B^A (Xdx + Ydy + Zdz), \tag{2}$$

the suffixes indicating the points between which the work is done.

The work done by a force whose point of application is displaced from any one position, A, to any other B, is often very

usefully represented graphically by means of a *Work Diagram.* If in any position P is the magnitude of the force, and dp the projection of the displacement of its point of application along the direction of P, the element of work is

$$P \, . \, dp,$$

and the whole work is the integral of this. Hence if we take two rectangular axes, Ox and Oy, and lay off, successively, along Ox the values of dp as they occur in the working of the force between A and B; and if perpendicularly to each of these elements we draw the corresponding value of P (as an ordinate), the extremities of these ordinates will trace out a curve whose area will represent the work done.

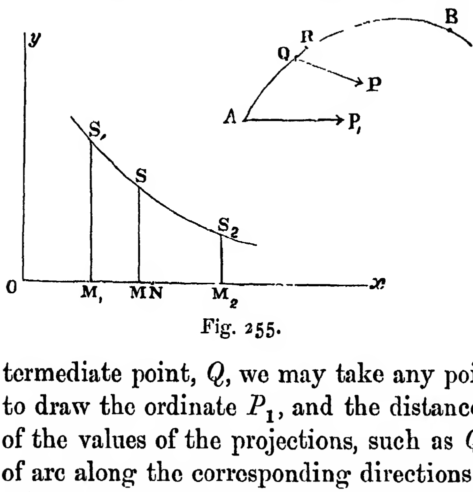

Fig. 255.

Thus, in Fig. 255, if P_1 is the magnitude of the working force at A, P_2 its magnitude at B, while P is its magnitude at any intermediate point, Q, we may take any point, M_1, on Ox at which to draw the ordinate P_1, and the distance $M_1 M$ will be the sum of the values of the projections, such as Qq, of the elements, QR, of arc along the corresponding directions of P between A and Q. We may, of course, choose the small arcs QR, ... of such lengths that the elements, MN, ... are all equal, i.e. dp may be taken as a constant element.

The expression $\int_B^A P dp$ for the work done between A and B becomes then

$$\text{the area } M_1 S_1 S_2 M_2,$$

properly translated from square centimetres (suppose) into kilogramme-metres, according to the scale of length on which force magnitude is represented in drawing the ordinates MS, and (generally) the diminished scale on which the projections Qq are represented by the elements MN.

If C. G. S. units are adopted, the unit of work is that done by a dyne in displacing its point of application through one centimetre in its own direction. This unit of work is called an *erg.*

Examples.

1. If one end of an elastic string is fixed while the other is drawn out through a given distance, find the work done by its tension, and the work diagram.

If l_0 is the natural length of the string, λ its modulus of elasticity, and l any stretched length which is productive of a tension T, we have $T = \lambda \frac{l - l_0}{l_0}$. For a small increment of length, dl, the tension does work equal to $-Tdl$; therefore disregarding the sign of the work, we may represent it by drawing the values of $l - l_0$ along Ox, so that OM is proportional to $l - l_0$; then at M we are to draw an ordinate, MS, proportional to T, and therefore proportional to OM. The locus of S is obviously a right line passing through O, and the work done by the tension for any amount of extension is represented by the area of a trapezium, affected with a negative sign.

The amount of work done by the tension in an extension from a length l_1 to a length l_2 is

$$-\frac{\lambda}{2l_0}(\overline{l_2 - l_0}^2 - \overline{l_1 - l_0}^2).$$

2. Another simple example of a work diagram is furnished by a gas enclosed in a cylinder fitted with a gas-tight piston, the gas expanding or contracting at a constant temperature.

In this case let us calculate the work done by the total pressure on the piston in the expansion of the gas by a given amount.

If P is the force exerted on the piston, and x, the distance of the piston, in any position, from the closed end of the cylinder, the law of Boyle and Mariotte gives

$$Px = \text{constant} = P_1 x_1,$$

where P_1 is the pressure in the first position and x_1 the distance of this position from the closed end.

The values of x being laid off along Ox, the extremities of the ordinates will trace out a rectangular hyperbola, and the area included between any portion of this curve, the ordinates at its extremities, and the axis of x, represents the work done by the pressure. The work done by the pressure from x_1 to x is

$$P_1 x_1 \log_e \frac{x}{x_1}.$$

3. In general, if a gas expands from a volume v_1 to a volume v_2, and if p is its intensity of pressure (or pressure per unit area), the work done by the gas against external resistance is

$$\int_{v_1}^{v_2} p\,dv. \tag{a}$$

For, if at any time the gas is enclosed within a surface S, whose element of area at any point is dS, the amount of pressure on this

element is pdS; and if in a small expansion the element dS is driven out along the normal through a distance dn, the work done by the pressure on dS is $pdS \cdot dn$; therefore for the small expansion of the whole volume enclosed by S the sum of the works done by the pressures on all its elements dS is (since p is constant throughout the gas), $p\int dSdn$; but $\int dSdn$ is the increase of volume of the whole gas for the small expansion considered, that is, dv; hence the work for this expansion is pdv, and therefore in the change from volume v_1 to volume v_2—the intensity of pressure, p, of course continuously varying—the work done is given by (a).

For example, if the gas changes *adiabatically*—i. e. so that no *heat* is conducted either into or out of it, while its *temperature* and intensity of pressure both vary—the relation between p and v is

$$pv^k = \text{constant}, \tag{b}$$

where k is about 1·408. In this case the curve whose abscissae and ordinates are the varying values of v and p is asymptotic to both axes—like the rectangular hyperbola

$$pv = \text{constant}, \tag{c}$$

which represents the relation between p and v when the expansion is unaccompanied by change of temperature—but it approaches the axis of volumes more rapidly than the hyperbola.

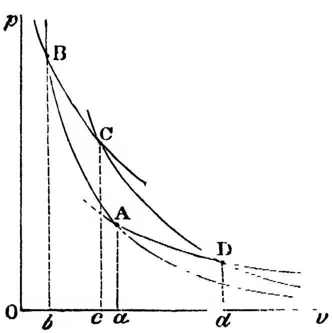

Fig. 256.

The curves obtained by varying the constant in (b) are called *adiabatics*, while those given by (c) are called *isothermals*. Thus, let A be a point whose co-ordinates Oa and aA are v_1 and p_1, respectively; then the curve whose equation is

$$pv^k = p_1 v_1^k$$

is AB, while the curve (rectangular hyperbola) whose equation is

$$pv = p_1 v_1$$

is AD. The co-ordinates of the points on AB between A and B represent the states of the gas as to volume and intensity of pressure in the adiabatic transformation from state A to state B.

A gas contained in a cylinder with a gas-tight piston can be transformed adiabatically and isothermally, successively, to any extent in the following manner. Suppose the base of the cylinder to be made of thin polished copper or silver. (Theoretically this base is to have perfect thermal conductivity, i. e. any heat applied to the outside is instantly transmitted to the inside, any difference of temperature between the outer and the inner surfaces of the base being at once annulled. Thin polished silver or copper will be an approximation. With such a base we are to imagine heat as flowing with no resistance into or out of the cylinder.)

Let the piston and all the rest of the cylinder be made of an infinitely bad thermal conductor, so that no heat can enter or leave the cylinder anywhere except through the base.

To produce adiabatic transformation. Place the cylinder with its base on a slab which is an infinitely bad thermal conductor, and do work on the gas by pressing down the piston. No heat can get into the cylinder by conduction from without, and none can leave it. Moreover, of the work thus done by the piston on the gas a portion goes to increase the energy of motion of its molecules, and the remainder is used in doing work against the (repulsive) forces existing between these molecules. From an experiment of Joule's, however, it appears that these molecular forces are non-existent; and subsequent experiments by Joule and Thomson show that, though this is not perfectly true for all gases, it is so nearly true, that the work absorbed in overcoming these molecular forces may be quite neglected.

The result, then, is that the work done on the gas goes wholly to increase its heat, and therefore its temperature. [Observe, this is not a contradiction of our supposition that no heat is communicated to it by conduction from any external source.]

If, instead of compression by means of the piston, the gas is allowed to expand and drive the piston before it, its temperature falls in an adiabatic transformation.

To produce isothermal transformation. Place the cylinder with its base on a very large reservoir of heat—so large that the volume of the gas is negligible in comparison—and let the temperature of the heat in the reservoir be the same as that of the heat of the gas. Allow the piston to be driven by the gas. The effect of even the smallest expansion would be a lowering of the temperature inside the cylinder, but as the base is an infinitely good conductor, the inequality of temperature inside and outside is instantly annulled by a flow inwards of heat from the reservoir, the temperature of which (on account of its capacity) suffers no sensible diminution. Thus the temperature inside the cylinder remains constant all through the expansion.

The piston might also be pressed down so as to compress the gas, the instantaneous effect being a rise of temperature, which is instantly annulled by the flow of heat from the gas into the reservoir.

The theoretical processes here described are those which are postulated in the working of *Carnot's Engine*, the theory of which is fundamental in Thermodynamics (see Clerk Maxwell's *Theory of Heat*, or almost any work on Physics).

Starting with the state represented in Fig. 256 by the point A, let the following cycle of operations occur:—adiabatic compression represented by the adiabatic AB, until state B is reached; isothermal expansion represented by BC, the gas receiving heat at constant temperature, and doing external work by driving the piston before it, until state C is reached; adiabatic expansion represented by CD,

the gas driving out the piston and doing external work, while its temperature falls and it receives no heat, until the temperature which it had originally (at A) is reached; finally, isothermal compression represented by DA, the piston being forcibly driven down until the original state (A) is reached.

It is required to calculate the whole amount of positive work done by the gas. This work is obviously the areal sum

$$-AabB+BCcb+CDdc-DdaA,$$

where a, b, c, d are the feet of the ordinates of A, B, C, D. Let the equation of

$$AD \text{ be } pv = m; \quad BC \text{ be } pv = m';$$
$$AB \text{ be } pv^k = n; \quad CD \text{ be } pv^k = n'.$$

Then the area $AabB = \frac{n}{k-1}\left(\frac{1}{v_2^{k-1}} - \frac{1}{v_1^{k-1}}\right)$, where v_1 and v_2 are the abscissae of A and B. But $v_2^{k-1} = \frac{n}{m'}$, and $v_1^{k-1} = \frac{n}{m}$; therefore this area $= \frac{m'-m}{k-1}$, which value is also that of $CDdc$. Hence the external work done by the gas is

$$\frac{m'-m}{k-1}\log_e\frac{n'}{n},$$

and this is also, of course, the area of the figure $ABCD$ included between the two isothermals and the two adiabatics.

272.] **Static Energy, or Potential Work of a Force System.** If the point of application of a force whose components are X, Y, Z occupies at any instant a position which we may denote by (p), and if (p_0) denotes any other position which it might occupy, the amount, Π, of work which the force can do in the displacement from (p) to (p_0) is given by the equation

$$\Pi = \int_{(p)}^{(p_0)} (Xdx + Ydy + Zdz). \tag{1}$$

The amount of work which the force can do from the present position (p) *to the supposed position* (p_0) *is called the* Potential Work *of the force.*

In the same way, if any number of forces act on any system of particles, m_1, m_2, ..., and if the present system of positions of these particles, or their present configuration, is denoted by (p), while another configuration, or system of positions which they might occupy, is denoted by (p_0), the whole amount of work

which the forces can do in the motion from the present to the contemplated position is given by the equation

$$\Pi = \Sigma \int_{(p)}^{(p_0)} (Xdx + Ydy + Zdz), \tag{2}$$

where Σ denotes a summation of the works done on all the particles. The configuration denoted by (p_0) may be taken arbitrarily. We shall speak of it as *the configuration of reference.* Here, as before, Π is *the potential work of the forces of the system.*

Defining the term *Energy* to mean *capacity for doing work*, we may speak of the Potential Work of a force system as its *Static Energy* *.

If the particles do not form a rigid body, but can alter their relative distances; and if, moreover, they exert on each other forces, either of attraction or of repulsion, the work done by the internal forces in the change of configuration must, of course, be included in the Static Energy of the system; so that if Π_i and Π_e are the potential works of its internal and external forces, respectively, the total Static Energy of the system is

$$\Pi_i + \Pi_e.$$

Any material system—whether it consists of particles at finite distances from each other, each acted upon by some external force and also by attractions from neighbouring particles, or particles at infinitesimal distances (as in the case of a bent spring, a membrane, or an elastic string)—may occupy several different configurations successively and finally return to its original configuration (p). If when it does return to its original configuration, the Static Energy of its force-system (internal and external forces included) returns also to its original value, the system is said to be *Conservative.* The consideration of such a system is of the greatest importance.

Any material system will be conservative when for any small derangement of the particles the work done by the external forces is the differential of a single-valued function of the co-ordinates of the particles, and the internal forces are functions only of the mutual distances of the particles, and are directed in the lines joining them.

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* It is usually spoken of as 'Potential Energy'—an illogical term which, as has been pointed out by an able writer, expresses 'a double remotion from actuality.'

For if the co-ordinates of the particles are (x_1, y_1, z_1), &c., and the external forces (X_1, Y_1, Z_1), &c., the work of the external forces for any small derangement is

$$X_1 dx_1 + Y_1 dy_1 + Z_1 dz_1 + X_2 dx_2 + Y_2 dy_2 + Z_2 dz_2 + \ldots,$$

and if this $= d\phi\ (x_1, y_1, z_1, x_2, y_2, z_2, \ldots)$, the Static Energy of these forces is $\phi_0 - \phi$, where ϕ_0 is the value of ϕ when the co-ordinates of the configuration (p_0) are substituted; and if ϕ is not a multiple-valued function—such as $\tan^{-1}\frac{y_1}{x_1}$—it is obvious that the Static Energy of the external forces must always be the same whenever the system has the same configuration.

Again, if the internal force between m_1 and m_2 is expressed as $f(r_{12})$, where r_{12} is the distance between them, and if it is directed in the line joining them, the element of work of this force is $\pm f(r_{12}) \,.\, dr_{12}$, according as the force is repulsive or attractive. Hence if $f(r_{12}) \,.\, dr_{12} = d\psi\,(r_{12})$, the Static Energy of the internal forces in any configuration is, by summation for all the particles,

$$\pm [\Sigma\psi(r)_0 - \Sigma\psi(r)],$$

which is manifestly the same whenever the configuration is the same.

For example, an elastic rod bent and twisted in any way, but not to such an extent as to alter sensibly its constants of elasticity, will be an example of a conservative system, if, moreover, the bending and twisting are not accompanied by heating. The effect of such heating might be to alter its various elastic constants in such a manner that, if it returns to its original configuration, the amount of work required to produce a given deformation either by bending or by twisting would not be the same as it was originally to produce exactly the same deformation.

If the deformation is produced slowly, the heating effect is avoided, and the system is conservative.

By definition, if work, W, is done by external agency on a conservative system to change its configuration from (p) to (p'), the system will give back exactly the same amount, W, of work against external resistance in returning from (p') to (p).

A simple example of a non-conservative system is furnished by a heavy particle on a rough inclined plane of inclination i.

To raise the particle through a given vertical height, h, by an up-plane force an amount of work equal to $wh\ (1+\mu \cot i)$ must be expended; while if the particle is allowed to slide down to its original position, it will give out only the amount $wh\ (1-\mu \cot i)$, and would give out none if $\mu =$ or $> \tan i$.

In all such cases—i. e. cases in which friction comes into play—a part of the work expended on the system in changing its configuration is transformed into heat, which is speedily lost to the system; and, in general, if any machine, or combination of machines, transforms a portion of the work done on it into heat, it cannot restore even so much of the work as has not been thus transformed, i. e. it is non-conservative.

273.] **Stability and Instability of Equilibrium.** When a rigid body, or any material non-rigid system, in equilibrium under the action of given forces is slightly disturbed from its position, it will not, in general, be in equilibrium in the new position. Now the effect of all the forces in play in the new position may be either to drive it back to the original position, or to deviate it still further. In the former case the equilibrium is *stable*, and in the latter *unstable*.

As an example for the case of a rigid body, suppose a heavy bar, AB, moveable round a smooth horizontal axis fixed through the end A. If the rod is placed in a vertical position, it will be in equilibrium; but if the end B is vertically *above* A, a slight displacement will cause the rod to fall from this position; while if the end B is *below* A, and the rod is slightly displaced, it will return to its position of equilibrium.

As an example for a non-rigid system, take the case of an indiarubber ring on an umbrella handle. If the substance of the ring is rotated round the circle formed by the centres of all its normal sections through an angle which is constant all through the ring, one configuration of equilibrium is obtained when this angle of rotation is π, i. e. when the ring is turned inside-out. But this configuration is, of course, unstable, the slightest disturbance causing the ring to return to its natural state. On the other hand, the natural state of the ring on the handle is a stable configuration of equilibrium.

274.] **Universal Criterion of Stability and Instability.** The determination of the nature of the equilibrium of any system—i. e. its stability or instability—is a question belonging to

Kinetics. The conditions as regards constraints and connexions of parts of the system with each other will enable us to express any possible configuration of the system in terms of a certain number of independent variables, $q_1, q_2, q_3, \ldots$, which may be described as 'co-ordinates' of the system, by an extension of the usual employment of this term. For example, suppose the system to consist of two particles, B and C, which are connected by an inextensible string, while another inextensible string, BA, is attached to B, and the system is suspended vertically by fixing the end A of the second string. In this case, supposing the displacements to be confined to a given vertical plane, if we imagine any configuration satisfying the conditions of the system, i.e. one in which the distances AB and BC are each constant, such a configuration is obtained by deviating AB from the vertical through any angle, θ, and then deviating BC from the vertical through any angle, θ', these two angles being entirely independent of each other. The configuration of the system, then, depends on the two independent variables θ and θ', which are its 'co-ordinates.'

If the displacements of the particles are not confined to any vertical plane, AB can be deviated through an angle θ from the vertical, and rotated (after the manner of a conical pendulum) round the vertical through an angle ϕ; and BC can be similarly displaced through angles θ' and ϕ'; so that there are *four* generalized co-ordinates $(\theta, \theta', \phi, \phi')$ of this system in the most general case of its displacement.

Such variables are usually called the *generalized co-ordinates* of the system, and they determine the number of degrees of freedom of the system—this being equal to the number of the generalized co-ordinates.

The kinetical process which determines whether the equilibrium of the system is stable or unstable consists in supposing each of the generalized co-ordinates, q, to receive any small increment, Δq, and then, from the equations of motion of the system, expressing each Δq as a function of the time. If the value of Δq is a periodic function of the time, the magnitude of Δq will oscillate between infinitely narrow limits, and the equilibrium of the system will be stable; while if any of the displacements Δq involves the time in a non-periodic form of the type e^t, this displacement increases indefinitely, and the equilibrium is unstable.

The result is this—*If for any possible small displacement of the system from its configuration of equilibrium there would be positive work done by the acting forces, both external and internal, the configuration is unstable; while if for every possible small displacement the sum total of the works of these forces is negative, the configuration is stable;* in other words, *the system will be in* **stable equilibrium** *if the* **Static Energy** *of the system, i. e. the* **Potential Work** *of its forces* (*internal and external*), *is a* **minimum**, *and in unstable equilibrium if this potential work is a maximum.*

This fundamental result we shall assume, referring the reader for the proof to Lagrange's *Mécanique Analytique*, 6th section of the *Dynamique*, p. 320; to Thomson and Tait's *Natural Philosophy*, Arts. 291, &c.; and to Laurent's *Traité de Mécanique Rationnelle*, vol. ii, p. 222, where an extremely concise proof by Dirichlet is given.

We shall revert to the proof of this principle in the next Article.

275.] **Work Coefficients.** When the rectangular co-ordinates (x_1, y_1, z_1), &c., of the points of application of the forces of the system are all independent, since

$$-d\Pi = X_1 dx_1 + Y_1 dy_1 + Z_1 dz_1 + X_2 dx_2 + \dots, \tag{1}$$

we see that the differential coefficient of the Potential Work (with sign changed) with respect to any co-ordinate is the corresponding component of force. Thus $-\dfrac{d\Pi}{dx_1} = X_1$, &c. But if the co-ordinates are not all independent, but expressible in terms of a number, k, of independent variables, $q_1, q_2, \dots q_k$, this is no longer true. Expressing the co-ordinates $x_1, y_1, z_1, \dots$ in terms of the q's, equation (1), for the element of Potential Work assumes the form

$$-d\Pi = Q_1 dq_1 + Q_2 dq_2 + \dots + Q_k dq_k, \tag{2}$$

in which the coefficients $Q_1, Q_2, \dots$ may be of the dimensions either of *force* or of *couple*, according to the nature of the generalized co-ordinates $q_1, q_2, \dots$. In all cases each term, $Q_1 dq_1$, in (2) is an elementary work, so that if q_1 is a *linear* co-ordinate, like x_1, the coefficient Q_1 will be of the dimension of *force*; but if q_1 is an *angle*, Q_1 will be of the dimension of *couple*.

Take, for example, the case of two coplanar forces, P_1 and P_2, acting at the ends, A and B, of a line of constant length, a, and

consider only displacements in the plane of the forces. The generalized co-ordinates of the system may be taken as the rectangular co-ordinates (x, y) of A, and the angle, θ, which AB makes with the axis of x. If (x', y') are the co-ordinates of B, we have $x' = x + a\cos\theta$; $y' = y + a\sin\theta$, and, the components of P_1 and P_2 being, respectively, (X_1, Y_1) and (X_2, Y_2), we have

$$-d\Pi = (X_1 + X_2)\,dx + (Y_1 + Y_2)\,dy + a(Y_2\cos\theta - X_2\sin\theta)\,d\theta,$$

in which the coefficients of dx and dy are of the dimensions of force, while that of $d\theta$ is of the dimensions of couple.

The coefficients $Q_1, Q_2, \ldots$ in (2) are sometimes spoken of as 'generalized components of force.' This expression is very objectionable on more grounds than one; but we fall into no error if we describe them as *Work Coefficients.* Thus Q_1 is the q_1-rate at which the system does work if the other independent variables, $q_2, \ldots q_k$, are all kept constant and q_1 alone allowed to vary; and it does not appear to be possible to specialize the meanings of the Q's any further—i.e. to give a rule applicable to all cases for localizing Q_1, $Q_2, \ldots$ as forces or couples at particular points or round particular axes in the system.

Since in a position of equilibrium $d\Pi$ is zero for all possible displacements, in such a position we must have

$$Q_1 = 0, \quad Q_2 = 0, \ldots Q_k = 0. \tag{3}$$

Now the fundamental principle of last Article, that the Potential Work of the system of forces, both internal and external, is a minimum in a configuration of stable, and a maximum in a configuration of unstable, equilibrium cannot be inferred from the vanishing of all the first differential coefficients $Q_1, Q_2, \ldots$. For, since Π is a function of several independent variables, k in number, there are $k-1$ additional independent conditions that Π should be either a maximum or a minimum.* In a particular case, however, the truth of the principle can be seen without the general kinetical investigation. This case is that in which the material system has one degree of freedom, i.e. when its position depends on a single variable, q. Here, since $\frac{d\Pi}{dq} = 0$ in the position of equilibrium, it follows that Π is, in general, either a maximum or a minimum; and it is easy to see that the maximum belongs to instability. For, if the equilibrium is unstable, the

* Williamson's *Diff. Cal.*, Note 2.

system will require positive work to be done on it by an external agent to resist the growth of the displacement dq; that is, the forces (internal and external) of the system must during the displacement be doing positive work—resisting the positive work which the external agent is applying; in other words, in leaving the position of equilibrium, the Static Energy of the given system is diminished. Clearly, then, the maximum value of Π corresponds to instability.

276.] **Maximum or Minimum height of the Centre of Gravity.** When gravity is the only external force, besides the reactions of smooth fixed surfaces, acting on a material system, and when for any change of its configuration its internal forces (such as mutual reactions at the contacts of smooth parts) do no work, the Potential Work of the forces is simply

$$W.\bar{z},$$

where W is the total weight of the system and $\bar{z}$ is the height of its centre of gravity above any horizontal plane which is assumed as the *reference position* (Art. 272) of the centre of gravity.

For, let $w_1, w_2, \ldots$ be the weights and $z_1, z_2, \ldots$ the heights of the centres of gravity of the various separate bodies, or particles, of the system. Then the virtual work of the system for any small displacements is $-w_1dz_1 - w_2dz_2 - w_3dz_3 \ldots$; hence*

$$d\Pi = w_1dz_1 + w_2dz_2 + \ldots = W.d\bar{z},$$

$$\therefore \quad \Pi = W.\bar{z},$$

the reference level being taken as that from which $\bar{z}$ is measured.

Now the maximum value of Π will occur when $\bar{z}$ is greatest; hence *when the centre of gravity of any system of bodies is in the lowest position that it can occupy consistently with the geometrical conditions of the system, that system is in a position of stable equilibrium; and when its centre of gravity is in the highest position, the system is in a position of unstable equilibrium.*

Unless the position of the system depends on a single variable, we cannot assert conversely that a position of equilibrium is one in which the height of the centre of gravity is either a maximum or a minimum.

If any bodies of the system rest on *rough* curves or surfaces,

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* This assumes that none of the geometrical forces required for a position of equilibrium are infinite; for the term $\lambda\delta L$ cannot be assumed to vanish, even though $\delta L = 0$, if λ is infinite.

the equation of virtual work will involve the reactions of these curves or surfaces for displacements along them. Hence we have no longer the equation $W.\delta\bar{z} = 0$, and the principle of maximum or minimum height of the centre of gravity does not hold.

Even when the position depends on one variable, it may happen that in a position of equilibrium the height of the centre of gravity is neither a maximum nor a minimum. Take, for example, the case of a heavy particle placed at a point of inflexion on a smooth curve in a vertical plane, the tangent at the point being horizontal. The particle is evidently in equilibrium, since for a small displacement $P\delta z$ is zero, P being the weight and z the height of the particle. But z is neither a maximum nor a minimum, and the equilibrium, accordingly, is stable for a small displacement along the upper part of the curve, and unstable for a displacement along the lower part.

When the system has only one degree of freedom, the centre of gravity describes, in all positions of the system compatible with the given conditions, a curve which is sometimes very easily found. In the position of equilibrium the centre of gravity will be the point of contact of a horizontal tangent to this curve, and in this manner we can most readily perceive the nature of the equilibrium of the body.

When the system has more than one degree of freedom, it may happen that its centre of gravity is constrained, in all displacements compatible with the connexions, to describe a fixed *surface.* In this case the position of equilibrium will be one in which the tangent plane to this surface at the centre of gravity is horizontal; and if the surface lies entirely below the tangent plane in the neighbourhood of the point of contact, the equilibrium will be unstable, as in the case of a curve; if the surface lies above the tangent plane, the equilibrium will be stable; and if the tangent plane intersects the surface in a real curve in the neighbourhood of contact, the equilibrium will be stable for some displacements and unstable for others.

277.] **Continuous Equilibrium.** If in all positions of the system, compatible with the geometrical conditions, the statical equation

$$d\Pi = 0$$

is satisfied, every position is one of equilibrium. Writing down this equation in all positions and adding, the left sides of the

equations thus obtained is evidently the same thing as integrating it. Hence if all positions of the system are positions of equilibrium, the applied forces must satisfy the equation

$$\Pi = \text{constant}.$$

In the particular case of a heavy system under the action of gravity alone, Π is $W.\bar{z}$; therefore if a system is continuously in equilibrium under the action of gravity, the centre of gravity of the system for all displacements compatible with the conditions moves in a fixed horizontal plane, or, in other words, *maintains a constant height.*

Examples.

1. A heavy beam, AB (Fig. 121, Art. 104), rests on two smooth inclined planes; find the nature of its equilibrium.

It is very easy to prove that if the right line AB moves between two fixed right lines, OA and OB, the given point G on AB describes an ellipse whose equation with reference to OA and OB as axes of x and y is

$$\frac{x^2}{b^2} + 2\frac{xy}{ab}\cos(\alpha+\beta) + \frac{y^2}{a^2} = 1.$$

The centre of this ellipse is the point O. In the position of equilibrium G is the point of contact of a horizontal tangent to this ellipse. Now two such tangents can be drawn, one above the intersection of the inclined planes and the other below it. There are, therefore, two positions of equilibrium; that with which we were concerned in the example of Art. 104 is obviously the position in which G is at a maximum height, and it is, therefore, *unstable*; the other requires the planes to be prolonged below their line of intersection, and as it also requires the reactions of the planes to assume impossible directions, it is physically impossible. It would, however, be possible if the planes were replaced by smooth fixed rods to which the extremities of the beam are attached by rings. The second position of equilibrium would then be *stable*.

The impossibility in a certain case of any position of equilibrium, except one of continuous contact with either plane, which has been signalized in Art. 104, is now easily explained. It occurs when the point of contact of the horizontal tangent to the ellipse locus of G falls underneath the plane (α) or the plane (β), so that it is not a possible position of G.

The problem may be solved by a purely analytical method. If z is the height of the centre of gravity of the beam, it will be easily found that in the position of equilibrium

$$\frac{d^2 z}{d\theta^2} = -\frac{\sin\alpha\sin\beta\cos\theta}{(a+b)\sin(\alpha+\beta)}\{(a+b)^2 + (a\cot\alpha - b\cot\beta)^2\}.$$

2. Two given points of a body rest in contact with two smooth inclined planes; show that the equilibrium of the body is unstable.

We know that if two vertices of a *given* triangle move along two fixed right lines, the locus of the third vertex is an ellipse whose centre is the intersection of the given lines.

Hence, if we consider a given triangle in the body to be formed by the centre of gravity and the two points which are in contact with the planes, we see that the locus of the centre of gravity is an ellipse whose centre is at the intersection of the inclined planes. Now in the position of equilibrium the centre of gravity is the point of contact of a horizontal tangent to this ellipse. Hence the only possible position of equilibrium is one in which the height of the centre of gravity is a maximum; therefore the equilibrium is unstable; and if, as explained in the last example, the point of contact of the tangent falls underneath either plane, the only position of equilibrium of the body is one of continuous contact with one of the planes. The student will find several particular examples of this problem in Walton's *Mechanical Problems* (pp. 164, &c.), where the solutions are analytical.

3. A heavy body has two plane surfaces, CP and CQ (Fig. 257), which rest against two smooth fixed pegs, P and Q, the line PQ making an angle with the horizon; show that the positions of equilibrium are determined by drawing horizontal tangents to a Limaçon.

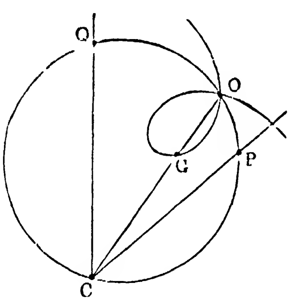

Fig. 257.

The centre of gravity and the pegs must lie in one vertical plane, which is that of the figure. Since P and Q are fixed points and the angle at C between the plane faces is constant, the circle described round the triangle PCQ is fixed in space. Again, let G be the centre of gravity of the body. Then since CG and CP are lines fixed in the body, the angle GCP is given; and if CG meets the circle in O, the point O is fixed in space; also the distance CG is given.

Hence in all positions of the body—i.e. in all positions of C on the circle—the centre of gravity is found by drawing the line OC from O to the circumference of the circle, and taking a constant length, CG, on this line. The curve deduced in this way from a circle is a Limaçon, which is, therefore, the locus of the centre of gravity.

A particular example has been already discussed on p. 166, vol. i.

4. A heavy plane body of any shape is suspended from a smooth peg, fixed in a vertical wall, by means of a string of given length, the extremities of which are attached to two fixed points in the body. Determine the nature of the equilibrium.

This problem, so far as the positions of equilibrium are concerned, has been already discussed (§ 107, Ex. 11, vol. i). We propose here

to show that there are two positions of stable and one position of unstable equilibrium. In the figure of the example referred to, the point of contact of GP_3 with the evolute is between G and P_3; the point of contact of GP_1 is between G and P_1; and the point of contact of GP_2 is on P_2G produced. Now it is easy to see that GP_3 is a line of maximum length drawn from G to the ellipse. For, let Q be a point on the ellipse close to P_3, and let QC be the normal at Q. Then C is the centre of curvature, and therefore the point of contact of GP_3 and the evolute. Hence $CP_3 = CQ$, therefore $GP_3 = GC + CQ$, which is $> GQ$, therefore $GP_3 > GQ$, and GP_3 is, therefore, a maximum.

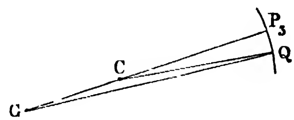

Fig. 258.

In the same way GP_1 is a maximum and GP_2 a minimum distance of G from the ellipse.

Hence, in the positions of equilibrium, GP_1 and GP_3 are maximum distances of the centre of gravity from the peg. The positions in which these lines are vertical are, therefore, positions of stable equilibrium. And since GP_2 is a minimum depth of G, the position in which GP_2 is vertical is one of unstable equilibrium.

5. To find the nature of the equilibrium of the beam in Example 7, p. 167, vol. i.

Take any position of the beam (in which, of course, the lines GW, AR, and PS (p. 148, vol. i) do not meet in a point). Then, if y is the ordinate of P, the point of contact of the beam and the curve, referred to a fixed horizontal axis, the ordinate of G will be

$$y + (GA - PA)\cos\theta,$$

or

$$y + a\cos\theta - x\cot\theta.$$

Denoting this by $\bar{y}$, we have

$$\frac{d\bar{y}}{d\theta} = \frac{dy}{d\theta} - a\sin\theta + \frac{x}{\sin^2\theta} - \cot\theta \cdot \frac{dx}{d\theta}.$$

Now $$\frac{dy}{dx} = \cot\theta, \quad \therefore \quad \frac{dy}{d\theta} - \cot\theta\frac{dx}{d\theta} = 0.$$

Hence $$\sin^2\theta\frac{d\bar{y}}{d\theta} = -a\sin^3\theta + x.$$

Differentiating this, and remembering that in the position of equilibrium $\frac{d\bar{y}}{d\theta} = 0$, we have

$$\sin^2\theta\frac{d^2\bar{y}}{d\theta^2} = \frac{dx}{d\theta} - 3a\sin^2\theta\cos\theta. \tag{1}$$

Again, since $\cot\theta = \frac{dy}{dx}$, we have

$$-\operatorname{cosec}^2\theta\frac{d\theta}{dx} = \frac{d^2y}{dx^2}.$$

But if ρ is the radius of curvature of the curve at P,

$$-\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} = \sin^3\theta\frac{d^2y}{dx^2}.$$

Therefore $\frac{d\theta}{dx} = \frac{1}{\rho\sin\theta}$, and (1) gives

$$\sin\theta\frac{d^2\bar{y}}{d\theta^2} = \rho - 3a\sin\theta\cos\theta$$
$$= \rho - 3PO.$$

Hence, since $\sin\theta$ is necessarily positive, $\frac{d^2\bar{y}}{d\theta^2}$ will be positive, and $\bar{y}$ therefore a minimum, if

$$\rho > 3PO.$$

The equilibrium will therefore be stable or unstable according as $\rho >$ or $< 3\ PO$.

To arrive at this result, it would have been sufficient to demonstrate it for a circle, which is very easily done. The curve in the neighbourhood of P may be replaced by the circle of curvature at this point.

6. Prove geometrically that the equilibrium of the beam in § 107, Example 2, vol. i, is stable.

7. Two uniform heavy rods freely jointed together at a common extremity rest on a smooth parabola whose axis is vertical and vertex upwards; find the position of equilibrium.

Ans. Let the weights of the rods be P and Q, their lengths $2a$ and $2b$, and let them make angles θ and ϕ, respectively, with the vertical in the position of equilibrium; then these angles are determined from the equations

$$Pa\sin^3\theta + (P+Q)\,m\cot\phi = 0,$$
$$Qb\sin^3\phi + (P+Q)\,m\cot\theta = 0,$$

$4m$ being the latus rectum of the parabola.

[Taking the tangent at the vertex as axis of y, the abscissa of the point of intersection of two tangents, $y = tx - \frac{m}{t}$ and $y = t'x - \frac{m}{t'}$, is $-\frac{m}{tt'}$. Hence

$$(P+Q)\bar{x} = Pa\cos\theta + Qb\cos\phi + (P+Q)\,m\cot\theta\cos\phi.$$

Then $\bar{x}$ is to be a maximum or minimum.]

8. A heavy uniform rod, AB, moveable about a fixed horizontal axis at A, has its end B connected with a string which, passing over a smooth pulley at a point C vertically above A, sustains a given weight which rests on a smooth inclined plane passing through C. Find the positions of equilibrium, and the nature of each.

Ans. Let W and $2a$ be the weight and length of the rod; P the weight on the plane whose inclination to the horizon is i; $2c$ the distance AC, and θ the inclination of the rod to the vertical. Then,

if $(c-a)\, W < 2\, Pc \sin i$, there will be three positions of equilibrium defined by the equations

$$\theta = 0,\ \cos\theta = \frac{W^2(a^2+c^2)-4\, P^2 c^2 \sin^2 i}{2\, ac\, W^2},\ \text{and}\ \theta = \pi.$$

The first and last positions are stable and the intermediate one is unstable.

If $(c-a)\, W > 2\, Pc \sin i$, there is no intermediate position, and the first and last positions are unstable and stable respectively.

9. One end of a beam rests against a smooth vertical plane, and the other on a smooth curve in a vertical plane; find the nature of the curve so that the beam may rest in all positions.

Ans. An ellipse whose axis major is the horizontal line described by the centre of gravity of the beam, the axis minor lying in the vertical plane.

10. A uniform heavy rod rests inside a smooth fixed sphere whose diameter is equal to the length of the rod. In all positions of the rod its centre of gravity is fixed; hence the rod should rest in all positions; but, except in the vertical position, it is impossible that the acting forces can give equilibrium. Explain this.

(See note, § 276.)

11. A uniform rod rests in all positions with its extremities on two smooth curves in a vertical plane; given the equation of one, find that of the other.

Ans. Let the axis of y be vertical, $2a$ the length of the rod, h the constant height of the centre of the rod, and $x = \phi\,(y)$ the equation of one curve; then the equation of the other will be

$$x = \phi(2h-y) - 2\sqrt{a^2-(h-y)^2}.$$

12. Find the general equation of a smooth curve (in a vertical plane) on which if the ends of a uniform rod are placed, the rod will rest in all positions.

Ans. If the line described by the centre of gravity is axis of x, the equation is the form $[\phi(y^2)+x]^2+y^2 = a^2$, where $2a =$ length of rod, and $\phi(y^2)$ is a function which does not change sign with y.

13. Investigate the equilibrium of the sphere and cone each resting on a smooth inclined plane, they being also in contact with each other, as in § 140, Example 5, vol. i.

Their positions being varied in any way, subject to the condition of contact, it is easy to prove that the locus of their common centre of gravity is a right line. If this line is not horizontal, it is impossible to have $d\bar{y} = 0$, and therefore, *in general*, there is no position of equilibrium in which each body is in contact with only one plane. If the line is horizontal, all positions are positions of equilibrium.

Taking horizontal and vertical lines through O as axes of x and y, respectively, and taking OA $(=\xi)$ as the single variable which determines the configuration of the system, we find that $(W+W')\,\bar{y}$ is the sum of a constant and the term

$$[W\sin\alpha - W'\frac{\sin\alpha'}{\cos\gamma}\cos(\alpha+\alpha'-\gamma)]\times\xi;$$

so that $\bar{y}$ will be constant if equation (3), in the example referred to, is satisfied.

14. Of all curves that can be drawn through two given points, A and B, and having the same length, determine that one whose revolution round the line AB generates a surface of maximum area.

Ans. A Common Catenary. For, imagine AB to be placed in a horizontal position, and let heavy uniform inextensible strings, all of the same length, coincide with various curves that can be drawn through A and B. These strings will one and all abandon their given configurations and become Catenaries. And since the equilibrium of the Catenary is stable, negative work would be done by all the forces acting on its particles for any imagined displacement of these particles which is consistent with the geometrical conditions of the figure. These conditions are simply that the two ends of the curve are fixed, and that there is perfect flexibility but no extensibility. Hence any change of figure consistent with these would *raise* the centre of gravity of the string; and therefore the centre of gravity of the Catenary is lower than the centre of gravity of any of the given curves; and since, by the Theorems of Pappus (vol. i, § 177), the surface generated by revolution is equal to the length of the revolving curve multiplied by the circumference of the circle described by its centre of gravity, the surface generated by the Catenary is greatest.

278.] Expansion of the Abscissa and Ordinate of a Curve in Powers of the Arc. Let A and B (Fig. 259) be any points on a curve, and let Am and An be the tangent and normal at A. Also let ψ be the angle between the normals at A and B, and let Am $(= x)$ and Bm $(= y)$ be the co-ordinates of B with reference to the tangent and normal at A as axes.

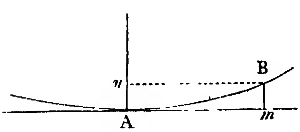

Fig. 259.

Then, by Maclaurin's Theorem, we have

$$\psi = \psi_0 + s\left(\frac{d\psi}{ds}\right)_0 + \frac{s^2}{1\,.\,2}\left(\frac{d^2\psi}{ds^2}\right)_0 + \ldots$$

s denoting the arc AB, and ψ_0, $\left(\frac{d\psi}{ds}\right)_0$, ..., the values of ψ and its differential coefficients at A.

Now $\psi_0 = 0$, and $\frac{d\psi}{ds} = \frac{1}{\rho}$, where ρ is the radius of curvature.

Hence

$$\psi = \frac{s}{\rho} + \frac{s^2}{1\,.\,2}\frac{d\left(\frac{1}{\rho}\right)}{ds} + \frac{s^3}{1\,.\,2\,.\,3}\frac{d^2\left(\frac{1}{\rho}\right)}{ds^2} + \ldots, \tag{1}$$

the suffix being omitted, it being understood that ρ is the radius of curvature at A.

Again, we have

$$x = x_0 + s\left(\frac{dx}{ds}\right)_0 + \frac{s^2}{1\,.\,2}\left(\frac{d^2x}{ds^2}\right)_0 + \dots;$$

also
$$\frac{d^2x}{ds^2} = -\frac{1}{\rho}\frac{dy}{ds}, \text{ and } \frac{d^2y}{ds^2} = \frac{1}{\rho}\frac{dx}{ds}.$$

But

$$\left(\frac{dx}{ds}\right)_0 = 1, \text{ and } \left(\frac{dy}{ds}\right)_0 = 0; \text{ therefore } \left(\frac{d^2x}{ds^2}\right)_0 = 0,\ \left(\frac{d^2y}{ds^2}\right)_0 = \frac{1}{\rho},$$

and the successive differential coefficients are calculated with ease.

We thus obtain

$$Bn = x = s - \frac{s^3}{6\rho^2} + \frac{s^4}{8\rho^3}\frac{d\rho}{ds} + \dots; \tag{2}$$

$$An = y = \frac{s^2}{2\rho} - \frac{s^3}{6\rho^2}\frac{d\rho}{ds} - \frac{s^4}{24}\left\{\frac{1}{\rho^3} - \frac{2}{\rho^3}\left(\frac{d\rho}{ds}\right)^2 + \frac{1}{\rho^2}\frac{d^2\rho}{ds^2}\right\} + \dots. \tag{3}$$

279.] **Equilibrium of a Heavy Body resting on a Fixed Rough Surface.** Let AD (Fig. 260) be a fixed rough surface on which a heavy body, AC, rests, under the action of gravity, at a single given point A; and let this body receive a slight displacement of rolling on the fixed surface.

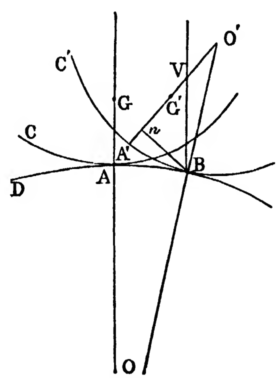

Fig. 260.

We propose to investigate the nature of the equilibrium. The figure represents a section of the bodies made by the vertical plane through their common normal, AO, in which the rolling takes place. We suppose the normal AO to be vertical.

Then, since in the position of equilibrium the body AC is acted on by only two forces—namely, its own weight and the total resistance of the fixed surface—its centre of gravity, G, must be vertically over the point of contact.

Let the point A of the rolling body come to A', and G to G', the new point of contact being B, and the new common normal OO'. Draw the vertical line BV, meeting $A'O'$ in V.

Then, if $A'V$ is $> A'G'$, the weight of the body acting through G' will produce a rotation round B which will send the body back to its original position; while, if $A'V$ is $< A'G'$, the rotation produced by the weight will be in the opposite direction, and the body will deviate still further from its original position. For stability, therefore,
$$A'V > A'G'. \qquad (1)$$

Let ρ and ρ' be the radii of curvature of the curves AD and AC at A, and let ψ and ψ' be the angles AOB and $A'O'B$. Then drawing Bn perpendicular to $A'O'$, we have
$$A'V = A'n + nV = A'n + Bn \cot A'VB\,;$$
but $\angle A'VB = \psi + \psi'$; therefore the condition for stability is
$$A'n + Bn \cot(\psi + \psi') > A'G'.$$
or, denoting $A'G'$ (or AG) by h,
$$Bn > (h - A'n)\tan(\psi + \psi'). \qquad (2)$$

Now, carrying approximations as far as s^3, we shall find from equation (1) of last Article that
$$\tan(\psi + \psi') = \left(\frac{1}{\rho} + \frac{1}{\rho'}\right) + \frac{s^2}{2}\left(\frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'}\right)$$
$$+ \frac{s^3}{6}\left\{\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} + 2\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)^3\right\},$$
s being the common length of the arcs AB and $A'B$.

Substituting this, and the values of Bn and $A'n$ from last Article, in (2), we get as the condition for stability
$$s - \frac{s^2}{6\rho'^2} > \left(h - \frac{s^2}{2\rho'} + \frac{s^3}{6\rho'^2}\frac{d\rho'}{ds'}\right)\left[\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)s + \frac{s^2}{2}\left(\frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'}\right)\right.$$
$$\left. + \frac{s^3}{6}\left\{\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} + 2\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)^3\right\}\right],$$
or
$$1 - \frac{s^2}{6\rho'^2} > h\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) + h\frac{s}{2}\left(\frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'}\right)$$
$$+ h\frac{s^2}{6}\left\{\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} + 2\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)^3\right\} - \frac{s^2}{2\rho'}\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) \dots \quad (3)$$

If we neglect all powers of s, the first condition of stability is

$$1 > h\left(\frac{1}{\rho} + \frac{1}{\rho'}\right),$$

or

$$h < \frac{\rho\rho'}{\rho+\rho'}. \tag{4}$$

If $h > \dfrac{\rho\rho'}{\rho+\rho'}$, the equilibrium will be unstable.

A special case occurs when $h = \dfrac{\rho\rho'}{\rho+\rho'}$, and this is commonly called the 'neutral' case, or the equilibrium is said to be neutral. We shall, however, call this the *critical* case.

To find the real nature of the equilibrium in this case, we revert to the general condition (3), and neglect all powers of s beyond the first. The condition for stability now is

$$0 > \frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'}.$$

Hence when $h = \dfrac{\rho\rho'}{\rho+\rho'}$, the equilibrium will be stable or unstable according as

$$\frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'} \text{ is negative or positive.} \tag{5}$$

The bodies are, however, frequently in contact at *vertices*, or points of maximum or minimum curvature, and then

$$\frac{d\frac{1}{\rho}}{ds} \text{ and } \frac{d\frac{1}{\rho'}}{ds'}$$

are both zero. Hence the condition (5) fails to determine the nature of equilibrium. Reverting to the condition (3), the terms as far as s^2 destroying each other on both sides, we see that equilibrium will be stable if

$$-\frac{1}{6\rho'^2} > \frac{h}{6}\left\{\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds^{2\prime}} + 2\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)^3\right\} - \frac{1}{2\rho'}\left(\frac{1}{\rho} + \frac{1}{\rho'}\right),$$

or, substituting $\dfrac{\rho\rho'}{\rho+\rho'}$ for h, if

$$\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} < -\frac{(\rho-\rho')(\rho+2\rho')}{\rho^3\rho'^2}; \tag{6}$$

and in the contrary case the equilibrium will be unstable.

If the lower surface is concave, instead of convex, to the upper, the conditions are obtained by changing the sign of ρ. Thus, the equilibrium will be stable or unstable, according as

$$h < \text{or} > \frac{\rho\rho'}{\rho-\rho'},$$

and in the critical case, the equilibrium will be stable or unstable, according as

$$\frac{d\frac{1}{\rho'}}{ds'} - \frac{d\frac{1}{\rho}}{ds}$$

is negative or positive; and in case of contact at vertices, the condition (6) is to be similarly modified.

If the body rest on a *plane* surface, $\rho = \infty$, and the differential coefficients of $\frac{1}{\rho}$ are all zero. Hence the limiting value of h for stability is ρ'; but if $h = \rho'$, the equilibrium will be stable or unstable according as $\frac{d\rho'}{ds'}$ is positive or negative; and if the point of contact is a vertex, equilibrium will be stable or unstable, according as

$$\frac{d^2\frac{1}{\rho'}}{ds'^2}$$

is negative or positive *.

* Different methods of arriving at the conditions for stability have been published in the *Quarterly Journal of Pure and Applied Mathematics* by Professor Curtis (vol. ix, p. 41), and Mr. Routh (vol. xi, p. 102). The kinetical method of treatment adopted by the latter is very exhaustive. The method in the text was employed independently by Professor Wolstenholme and the author.

It may be well to caution the student against the error of replacing the sections, AD and AC, of the surfaces in contact by their osculating circles at A. For, if we do this, the condition (5) necessarily disappears, and the application of (6) is not allowable, since, to the third power of the arc, the value of $A'n$ is not the same for the circle of curvature as for the curve AC, as at once appears from the expression for $A'n$ given by equation (3) of last Article. The nature of the equilibrium, therefore, as determined from the osculating circles is erroneous.

Examples.

1. If a cone of the same substance and of equal base with a hemisphere be fixed to the latter, so that their bases coincide, find the greatest height of the cone in order that the equilibrium may be stable, when the hemisphere rests symmetrically on a horizontal plane. (Walton's *Mechanical Problems*, p. 185.)

Ans. The height of the cone must be $< r\sqrt{3}$, r being the radius of the hemisphere.

2. Prove that any body with a plane base, resting on a fixed rough spherical surface, will, when the height of its centre of gravity has the critical value, be in unstable equilibrium.

3. A heavy body whose section in the plane of displacement is a catenary, resting on a rough horizontal plane, has its centre of gravity at the critical height; prove that the equilibrium is really stable.

(The condition (6) reduces in this case to $\dfrac{d^2\frac{1}{\rho'}}{ds'^2} < 0$ for stability.)

4. A heavy body in the shape of a paraboloid of revolution, placed on a rough horizontal plane, has its centre of gravity at the critical height; determine this height, and find the real nature of the equilibrium.

Ans. The critical height = the radius of curvature of the generating parabola at the vertex, and the equilibrium is really stable.

5. In the critical case, if both of the conditions (5) and (6) fail, prove that the equilibrium will be stable or unstable, according as

$$\frac{d^3\frac{1}{\rho}}{ds^3} + \frac{d^3\frac{1}{\rho'}}{ds'^3} - \left(\frac{1}{\rho} + \frac{1}{\rho'}\right)\left(\frac{4}{\rho} + \frac{1}{\rho'}\right)\frac{d\frac{1}{\rho'}}{ds'}$$

is negative or positive, the surfaces being convex towards each other.

6. A uniform heavy bar, AB, moveable in a vertical plane round a fixed smooth axis passing through A has a string attached to the end B; this string passes over a fixed pulley C vertically over A. Find the positions of equilibrium, and determine whether they are stable or unstable.

Ans. Let W = weight of bar, $2a$ its length, P = suspended weight, $AC = h$, $\theta = \angle CAB$. Then the positions of equilibrium are given by the equations

$$\theta = 0, \quad \cos\theta = \frac{a}{h} + \left(\frac{1}{4} - \frac{P^2}{W^2}\right)\frac{h}{a}, \quad \text{and} \quad \theta = \pi.$$

The first will be stable if $\dfrac{2h}{h-2a} > \dfrac{W}{P}$, and then the second (when it exists) will necessarily be unstable and the third stable. If the second does not exist, the third will be opposite in nature to the first.

[To find the condition for stability in this problem, we may either take any position of the bar and calculate the moment of force tending to turn it round A, or find the positions of the system for which the common centre of gravity of the bar and weight is highest or lowest. Employing the first method, if $M =$ the restoring moment, and $\phi = \angle ACB$, we see that

$$M = Ph \sin\phi - Wa \sin\theta. \tag{1}$$

Also
$$h \sin\phi = 2a \sin(\theta + \phi). \tag{2}$$

Now $M = 0$ in a position of equilibrium; and if $\frac{dM}{d\theta}$ is positive, a slight increase of θ will call into play a moment tending to restore equilibrium.

In the position $\theta = 0$, we have from (2)

$$\frac{d\phi}{d\theta} = \frac{2a}{h - 2a};$$

and from (1)
$$\frac{dM}{d\theta} = Ph\frac{d\phi}{d\theta} - Wa.$$

Therefore, &c.]

7. If the equilibrium in the first position is critical, find its real nature.

Ans. It is really unstable.

[In the position $\theta = 0$, it will be found from (2) that $\frac{d^2\phi}{d\theta^2} = 0$,

$$\frac{d^3\phi}{d\theta^3} = -\frac{2ah(h+2a)}{(h-2a)^3}; \quad \frac{d^2M}{d\theta^2} = 0, \quad \frac{d^3M}{d\theta^3} = -.]$$

8. Determine whether the equilibrium of the beam in § 105, Ex. 9, vol. i, is stable or unstable.

Ans. Unstable. [Either by taking the restoring moment about O, or by the maximum or minimum value of the static energy.

If we imagine the position in which the beam lies horizontal as the reference position, the acting forces, W and P, could do an amount of work equal to

$$Wa \sin\theta - P\{a + b - (a+b)\cos\theta\}$$

in reaching this position by a slipping of the ends of the beam along the planes. This is therefore the value of Π, the static energy—in which, if we please, we may discard the constant term $P(a+b)$. Therefore, &c.]

9. Four bars, AB, BC, CD, DA (§ 120, Ex. 4, vol. i), forming a plane quadrilateral, and freely jointed at the vertices, are kept in equilibrium by an elastic string stretched between the middle points of BC and DA, and an elastic strut compressed between the middle points of AB and CD, the string and the strut both following Hooke's Law. Show that there are always two, and there may be four, configurations of equilibrium.

282.] **Equations of Condition of Continuous Systems.** If the system of particles whose equilibrium is under consideration is continuous—as, for instance, an inextensible string, an inextensible membrane, or a rigid solid—the equations of condition will express the invariability of an infinitesimal element, such as the distance between two indefinitely close points.

Take, for example, the case of an inextensible string of which PQ (Fig. 261) is an elementary length, equal to ds. The equations $L_1 = 0$, $L_2 = 0, \ldots$ which express the invariable connexions of the particles of the system, will be $ds_1 =$ constant, $ds_2 =$ constant, ..., where ds_1, ds_2, ... are the distances between successive points on the curve; and the typical term $\lambda \delta L$ which enters into the equation of Virtual Work will be the typical term $\lambda \delta ds$.

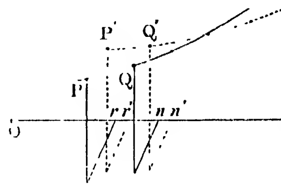

Fig. 261.

Let us inquire more particularly into the meaning of the expression δds. If we contemplate any small displacement whatever of the string, such that the element PQ comes into the position $P'Q'$, the new length $P'Q'$ being either greater or less than PQ, the meaning of the expression δds is $P'Q'-PQ$; and the condition that no change of length of the element takes place in the displacement is

$$\delta ds = 0.$$

Now (x, y, z) being the co-ordinates of P, we imagine these to receive, respectively, increments δx, δy, δz; i.e. the co-ordinates of P' are $(x+\delta x,\ y+\delta y,\ z+\delta z)$; while those of Q are

$$(x+dx,\ y+dy,\ z+dz).$$

The co-ordinates, therefore, of Q' (to which Q is imagined to be displaced) are represented by

$$x+dx+\delta(x+dx);\ y+dy+\delta(y+dy);\ z+dz+\delta(z+dz).$$

The excesses of the co-ordinates of Q' over those of P' are therefore $\quad dx+\delta(dx);\ dy+\delta(dy);\ dz+\delta(dz);$

and the length of $P'Q'$ being $PQ+\delta(ds)$, or $ds+\delta ds$, we have

$$(ds+\delta ds)^2 = (\delta x+\delta dx)^2+(dy+\delta dy)^2+(dz+\delta dz)^2;$$

or $$\delta ds = \left(\frac{dx}{ds}\frac{\delta dx}{ds}+\frac{dy}{ds}\frac{\delta dy}{ds}+\frac{dz}{ds}\frac{\delta dz}{ds}\right)ds, \qquad (\alpha)$$

neglecting infinitesimals of the fourth order, such as $(\delta ds)^2$, &c.

But the increments d and δ being completely independent and essentially distinguished as above explained, it is easy to see that the order in which the double operation $d\delta$ is performed on any function is indifferent; i.e. $d(\delta V)$ is precisely the same as $\delta(dV)$, where V is any function. In fact an inspection of the figure (Fig. 261) at once shows that $\delta(dx) = d(\delta x)$, the line Ox being the axis of x. For, let the abscissæ of P and Q be Or and On, those of P' and Q' being Or' and On', measured along Ox.

Then if x is the co-ordinate of P, $dx = rn$, and $\delta x = rr'$.

Also $\delta(dx)$ = value of dx in the new position − value of dx in old position $= r'n' - rn$; and $d(\delta x)$ = value of δx for Q − value of δx for $P = nn' - rr'$. But obviously

$$r'n' - rn = nn' - rr';$$

therefore $\delta(dx) = d(\delta x)$.

In virtue, then, of this commutative property of d and δ, (α) may be written

$$\delta\, ds = \left(\frac{dx}{ds}\frac{d\delta x}{ds} + \frac{dy}{ds}\frac{d\delta y}{ds} + \frac{dz}{ds}\frac{d\delta z}{ds}\right) ds. \qquad (\beta)$$

283.] **Variation of any Function. Particular Cases.** Since a variation of any function of the co-ordinates of a point consists in making infinitesimal increments to the several co-ordinates, it is clear that all the resulting changes are subject to the ordinary rules of the Differential Calculus. To fix ideas by means of an elementary example, suppose that we have a series of points lying on a circle whose equation is

$$x^2 + y^2 - a^2 = 0.$$

If now we imagine each point (x, y) on the circle displaced to an infinitely near position which is defined by adding to the abscissa a quantity equal to $\epsilon \,.\, y \sin\frac{x}{b}$, and to the ordinate a quantity $\epsilon \,.\, x \sin\frac{y}{c}$, where ϵ is an infinitely small quantity, we shall obtain a new curve differing infinitely little in position and shape from the original. In this particular case the increments which we have denoted by δx and δy are given by the equations $\delta x = \epsilon \,.\, y \sin\frac{x}{b}$, $\delta y = \epsilon \,.\, x \sin\frac{y}{c}$; and so in general, whatever be the laws according to which the variations are made.

It is obvious, then, that if u and v are any two functions of the co-ordinates of a point,

$$\delta \frac{u}{v} = \frac{v\delta u - u\delta v}{v^2}.$$

So, again, if V is any function of x, we have

$$\delta V = \frac{dV}{dx}\delta x,$$

any arbitrary change, δx, being made in x; and in passing to an adjacent point on a given curve or surface,

$$d(\delta V) = \frac{d^2 V}{dx^2}\delta x\, dx + \frac{dV}{dx} d(\delta x).$$

Also in an integration along a curve or surface, since this integration consists merely in a summation with respect to all the points on the curve or surface, we have

$$\delta \int V dx = \int \delta (V dx).$$

If, in particular, an integration, $\int V ds$, is performed along a curve, and all the points of the curve receive displacements such that the distance, ds, between two consecutive points remains unaltered, we shall have

$$\delta \int V ds = \int (\delta V) \, . \, ds;$$

and the same equation holds, in like case, if the integration is performed over a surface or throughout a solid if for ds we put the element of superficial area or the element of volume.

In this case also

$$\delta \frac{dx}{ds} = \frac{d\delta x}{ds}; \quad \delta \frac{d^2 x}{ds^2} = \frac{d^2 \delta x}{ds^2}; \quad \delta \frac{d^n x}{ds^n} = \frac{d^n \delta x}{ds^n}.$$

EXAMPLE.

Every element of mass of a solid is multiplied by the product of two of its co-ordinates, xy, and the sum of all such products ("product of inertia") taken. If the body receives a small displacement of rotation round the axis of z, find the variation of this sum.

Let dm be the element of mass at the point x, y, z; then the sum $= \int xy\, dm$. Now $\delta \int xy\, dm = \int \delta(xy) \, . \, dm = \int (x\delta y + y\delta x) dm$. But if the angle of rotation is $\delta\theta$, we have $\delta x = -y\delta\theta$, $\delta y = x\delta\theta$. Hence the variation of the sum is

$$\delta\theta \times \int (x^2 - y^2)\, dm.$$

To determine the variation of the angle between two consecutive tangents to any curve.

Let the tangents be at points, P, Q, separated by an arc of length ds, and let $d\theta$ be the angle between them. Then

$$d\theta = \frac{ds}{\rho}, \tag{1}$$

where ρ is the radius of absolute curvature of the curve. Now $\delta d\theta$ is what we have to find; and we shall suppose for generality that in the displacements of P and Q the length ds is altered. We have then

$$\delta d\theta = \frac{1}{\rho}\delta ds - \frac{1}{\rho^2} ds\,\delta\rho. \tag{2}$$

But
$$\frac{1}{\rho^2} = \frac{(d^2x)^2 + (d^2y)^2 + (d^2z)^2}{ds^4}; \tag{3}$$

$$\therefore\ -\frac{1}{\rho^3}\delta\rho = \frac{d^2x d^2\delta x + d^2y\, d^2\delta y + d^2z\, d^2\delta z}{ds^4} - 2\frac{(d^2x)^2 + (d^2y)^2 + (d^2z)^2}{ds^5}\delta ds;$$

$$\therefore\ -\frac{1}{\rho^2}\delta\rho = \rho\left(\frac{d^2x}{ds^2}\frac{d^2\delta x}{ds^2} + \frac{d^2y}{ds^2}\frac{d^2\delta y}{ds^2} + \frac{d^2z}{ds^2}\frac{d^2\delta z}{ds^2}\right) - \frac{2}{\rho}\frac{d\delta s}{ds}. \tag{γ}$$

Hence

$$\delta d\theta = \rho\left(\frac{d^2x}{ds^2}\frac{d^2\delta x}{ds^2} + \frac{d^2y}{ds^2}\frac{d^2\delta y}{ds^2} + \frac{d^2z}{ds^2}\frac{d^2\delta z}{ds^2}\right)ds - \frac{1}{\rho}\delta ds \tag{δ}$$

$$= \Big[-\frac{1}{\rho}\frac{dx}{ds}\frac{d\delta x}{ds} - \frac{1}{\rho}\frac{dy}{ds}\frac{d\delta y}{ds} - \frac{1}{\rho}\frac{dz}{ds}\frac{d\delta z}{ds}$$
$$+\rho\frac{d^2x}{ds^2}\frac{d^2\delta x}{ds^2} + \rho\frac{d^2y}{ds^2}\frac{d^2\delta y}{ds^2} + \rho\frac{d^2z}{ds^2}\frac{d^2\delta z}{ds^2}\Big]ds. \tag{ϵ}$$

To find the variation of the angle between two consecutive osculating planes of any tortuous curve.

[A *tortuous curve*, called also a 'curve of double curvature,' is one whose osculating plane varies from point to point.]

If l, m, n are the direction-cosines of the binormal, i.e. the perpendicular to the osculating plane, at any point of the curve, we have

$$l = \rho\left(\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}\right);\ m = \ldots;\ n = \ldots,$$

and if $d\phi$ is the angle between two consecutive osculating planes,

since the tangent line to the curve is perpendicular to two consecutive binormals, we have $\frac{dx}{ds} = \frac{mdn - ndm}{d\phi}$. Hence

$$d\phi = \rho\left(l\frac{d^3x}{ds^3} + m\frac{d^3y}{ds^3} + n\frac{d^3z}{ds^3}\right).ds,$$

and we shall find that

$$\delta d\phi = \Big[A_1\frac{d\delta x}{ds} + B_1\frac{d\delta y}{ds} + C_1\frac{d\delta z}{ds} + A_2\frac{d^2\delta x}{ds^2} + B_2\frac{d^2\delta y}{ds^2}$$

$$+ C_2\frac{d^2\delta z}{ds^2} + A_3\frac{d^3\delta x}{ds^3} + B_3\frac{d^3\delta y}{ds^3} + C_3\frac{d^3\delta z}{ds^3}\Big].ds, \qquad (\zeta)$$

where A_1, &c., are certain functions of the differential coefficients $\frac{dx}{ds}$, &c.

For any arbitrary displacement of a surface, $z = \phi(x, y)$, to find the variations of the partial differential coefficients $\frac{dz}{dx}$ and $\frac{dz}{dy}$. The arbitrary changes in x, y, z which we have hitherto denoted by δx, δy, δz we shall now find it convenient to denote by u, v, w, respectively.

Let P be any point (x, y, z) on a given surface—which surface we may, to fix ideas, imagine to be a thin sheet of india-rubber—whose points may receive, or be imagined to receive, any small displacements whatever. If these displacements are completely unhampered, any small element of area described round P on the undisplaced surface will be found on the displaced surface in a distorted form, and with its area altered in magnitude.

Suppose that Q is any point on the undisplaced surface indefinitely close to P, the co-ordinates of Q being $(x+\xi, y+\eta, z+\zeta)$. Then since z is determined when x and y are given (which would not be the case if instead of a *surface* we had a *solid* to deal with), the displacements u, v, w will each be some assigned function of x and y, i. e.

$$u = f_1(x, y);\ v = f_2(x, y);\ w = f_3(x, y). \qquad (\eta)$$

Let P' and Q' be the displaced positions of P and Q; and observe that $\frac{dz}{dx}$ means the increment of z divided by that of x, as we pass from a point P to a close point, R, such that P

and R have the same y. Imagine, then, Q to be so chosen that Q' and P' have the same y, so that the new

$$\frac{dz}{dx} = \frac{z \text{ of } Q' - z \text{ of } P'}{x \text{ of } Q' - x \text{ of } P'}.$$

Now the x of Q' is $x+u+\xi+f_1(x+\xi, y+\eta)$, according to the law expressed by equations (η); and this is

$$x+u+\xi+\xi\frac{du}{dx}+\eta\frac{du}{dy}.$$

Similarly the z of Q' is

$$z+w+\zeta+\xi\frac{dw}{dx}+\eta\frac{dw}{dy};$$

and since the y of Q' = the y of P', we have

$$\xi\frac{dv}{dx}+\eta\left(1+\frac{dv}{dy}\right)=0. \qquad (\theta)$$

Denoting, as is usual, $\frac{dz}{dx}$ by p and $\frac{dz}{dy}$ by q, and the values of these at P' by $p+\Delta p$ and $q+\Delta q$, we have

$$p+\Delta p = \frac{\xi\frac{dw}{dx}+\eta\frac{dw}{dy}+\zeta}{\xi\left(1+\frac{du}{dx}\right)+\eta\frac{du}{dy}}. \qquad (\iota)$$

Now since on the undisplaced surface $dz = pdx + qdy$, we have $\zeta = p\xi + q\eta$. Substitute this value in (ι), and then for $\xi:\eta$ put the value given by (θ), and we have, by neglecting such infinitesimals of the second order as the products $\frac{du}{dx}\frac{dv}{dy}$, &c.,

$$p+\Delta p = \frac{p\left(1+\frac{dv}{dy}\right)+\frac{dw}{dx}-q\frac{dv}{dx}}{1+\frac{du}{dx}+\frac{dv}{dy}};$$

$$\therefore \Delta p = \frac{dw}{dx} - p\frac{du}{dx} - q\frac{dv}{dx}. \qquad (\kappa)$$

Similarly, $$\Delta q = \frac{dw}{dy} - p\frac{du}{dy} - q\frac{dv}{dy}. \qquad (\lambda)$$

Examples.

1. Find the conditions to be satisfied by the displacements of all points on a perfectly inextensible surface.

The length of the line PQ must be unaltered whatever point Q may be. Now from the preceding we have

$$P'Q'^2 = [(1+\frac{du}{dx})\xi + \frac{du}{dy}\cdot\eta]^2 + [\frac{dv}{dx}\cdot\xi + (1+\frac{dv}{dy})\eta]^2$$
$$+[(p+\frac{dw}{dx})\xi + (q+\frac{dw}{dy})\eta]^2.$$

Hence the conditions for perfect inextensibility are

$$\frac{du}{dx} + p\frac{dw}{dx} = 0, \quad \frac{dv}{dy} + q\frac{dw}{dy} = 0,$$
$$\frac{du}{dy} + \frac{dv}{dx} + p\frac{dw}{dy} + q\frac{dw}{dx} = 0.$$

2. From these conditions find equations for the separate components of displacement.

Ans. The value of w is to be obtained from the partial differential equation

$$t\frac{d^2w}{dx^2} - 2s\frac{d^2w}{dxdy} + r\frac{d^2w}{dy^2} = 0,$$

where $$r = \frac{dp}{dx}, \quad t = \frac{dq}{dy}, \quad s = \frac{dp}{dy} = \frac{dq}{dx}.$$

In the case of a plane surface, $z = ax+by+c$, we find

$$u + aw = my + n; \quad v + bw = -mx + n',$$

where m, n, n' are arbitrary constants.

284.] **Equilibrium of an Inextensible String.** We now apply the method of Lagrange to determine the equations of equilibrium of an inextensible string acted on by any system of forces. Let, as previously, m denote the mass per unit length at any point of the string, and X, Y, Z the components of the external force, per unit mass, at the point.

Now the equations, $L_1 = 0$, $L_2 = 0, \ldots$ of condition are in this case $ds_1 =$ const., $ds_2 =$ const.,... and the general equation of equilibrium of Art. 260 becomes

$$m_1(X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1)ds_1 + m_2(X_2\delta x_2 + Y_2\delta y_2 + Z_2\delta z_2)ds_2 + \ldots$$
$$+\lambda_1\delta ds_1 + \lambda_2\delta ds_2 + \ldots \qquad = 0, \qquad (1)$$

the string being supposed to have assumed its position of equilibrium; for it is when the equilibrium position is assumed that the forces satisfy the above equation of Virtual Work.

Now the particles being infinitely numerous, we may write the above equation simply

$$\int m(X\delta x+Y\delta y+Z\delta z)ds-\int\lambda\delta ds=0. \tag{2}$$

Reducing all the variations to variations of x, y, z, or, in other words, substituting here the value of δds given in equation (β) of Art. 282, we have

$$\int\left[m(X\delta x+Y\delta y+Z\delta z)ds+\lambda\left(\frac{dx}{ds}d\delta x+\frac{dy}{ds}d\delta y+\frac{dz}{ds}d\delta z\right)\right]=0. \tag{3}$$

Now $\displaystyle\int\lambda\frac{dx}{ds}d\delta x=\left(\lambda\frac{dx}{ds}\delta x\right)_1-\left(\lambda\frac{dx}{ds}\delta x\right)_0-\int\delta x\frac{d}{ds}\left(\lambda\frac{dx}{ds}\right)\cdot ds,$

by integration by parts, the term $\left(\lambda\frac{dx}{ds}\delta x\right)_1$ being the value of $\lambda\frac{dx}{ds}\delta x$ at one of the limits of integration, i.e. at one extremity of the string; and $\left(\lambda\frac{dx}{ds}\delta x\right)_0$ being its value at the other extremity.

If we perform similar integrations for the other terms, (3) becomes

$$\begin{aligned}\lambda_1\left(\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y+\frac{dz}{ds}\delta z\right)_1-\lambda_0\left(\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y+\frac{dz}{ds}\delta z\right)_0 \\ +\int\left[\left\{mX-\frac{d}{ds}\left(\lambda\frac{dx}{ds}\right)\right\}\delta x+\left\{mY-\frac{d}{ds}\left(\lambda\frac{dy}{ds}\right)\right\}\delta y\right. \\ \left.+\left\{mZ-\frac{d}{ds}\left(\lambda\frac{dz}{ds}\right)\right\}\delta z\right]=0.\end{aligned} \tag{4}$$

Now, as in the equation of Art. 269 we equated to zero the coefficients of δx_1, δy_1, $\delta z_1,\ldots$, so here we have to put the coefficients of δx, δy, and δz equal to zero for each particle of the string; that is, we put the coefficients of these quantities under the sign of integration equal to zero. Hence we have at all points

$$\left.\begin{aligned}mX-\frac{d}{ds}\left(\lambda\frac{dx}{ds}\right)=0,\\ mY-\frac{d}{ds}\left(\lambda\frac{dy}{ds}\right)=0,\\ mZ-\frac{d}{ds}\left(\lambda\frac{dz}{ds}\right)=0,\end{aligned}\right\} \tag{A}$$

which equations are precisely the same as those of Art. 190; and it appears either by comparison of both sets of equations,

or by the end of Art. 269, that λ in these equations is minus the tension of the string.

The conditions of equilibrium, then, as expressed in (4), consist of two parts—namely, terms which relate to the extremities of the string (which are the terms outside the sign of integration), and terms which relate to every intermediate point in the string (which give the general equations of equilibrium above).

Equating to zero the terms outside the integral sign, we have

$$\lambda_1\left(\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y+\frac{dz}{ds}\delta z\right)_1-\lambda_0\left(\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y+\frac{dz}{ds}\delta z\right)_0=0. \quad (5)$$

Now, if the extremities of the string are fixed, they will be fixed in the displaced string, and every term of (5) vanishes, since

$$\delta x_1=\delta y_1=\delta z_1=\delta x_0=\delta y_0=\delta z_0=0.$$

But if each end is perfectly free, since δx_1, δy_1 ... are quite arbitrary and independent, we must have

$$\lambda_1=0 \text{ and } \lambda_0=0,$$

i. e. each terminal tension must be zero.

If the extremity $(x_1\ y_1\ z_1)$ is constrained to lie on a fixed surface, whose equation is $u=0$, we have the displacements of this extremity connected by the equations

$$\left(\frac{dx}{ds}\right)_1\delta x_1+\left(\frac{dy}{ds}\right)_1\delta y_1+\left(\frac{dz}{ds}\right)\delta z_1=0,$$

$$\left(\frac{du}{dx}\right)_1\delta x_1+\left(\frac{du}{dy}\right)_1\delta y_1+\left(\frac{du}{dz}\right)_1\delta z_1=0,$$

which give by the method of undetermined multipliers

$$\frac{\left(\frac{dx}{ds}\right)_1}{\left(\frac{du}{dx}\right)_1}=\frac{\left(\frac{dy}{ds}\right)_1}{\left(\frac{du}{dy}\right)_1}=\frac{\left(\frac{dz}{ds}\right)_1}{\left(\frac{du}{dz}\right)_1},$$

the geometrical meaning of which is that the direction of the string at this extremity is normal to the surface of constraint.

If the extremity is constrained to a curve whose equations are $u=0$, $v=0$, we find in the same way that at this extremity the direction of the string must be at right angles to the curve.

The method which we have just employed is the second method of Art. 183, and expresses that *the variation of the whole*

potential work of the external forces is zero, consistently with the geometrical condition that the distance between every two indefinitely close points in the string remains absolutely unchanged in the displaced position.

285.] **Equilibrium of an Extensible String.** In this case there are no geometrical conditions to be satisfied in the displacement (or deformation) of the string. Then the equation of equilibrium will simply express the condition that in the position of equilibrium the variation of the whole potential work of applied and internal forces is zero.

Now if we consider any elementary mass, $m\,ds$, whose length is ds, and whose internal force (the tension) is T, the work done by this force for a variation $\delta\,ds$ of the elementary length is (see Art. 70)

$$-T\delta\,ds.$$

Adding together the similar terms for all the elementary masses, we find that the variation of the potential work of the applied and internal forces is

$$\int m(X\delta x+Y\delta y+Z\delta z)ds-\int T\delta\,ds,$$

which differs from (2) only in having $-T$ instead of λ. Hence the whole discussion is exactly the same as before, and the results are those arrived at in Chap. XII.

There is, however, this distinction between the case of the elastic and that of the inelastic string—that in the second case the value of m, the density, is known at each point, since it can alter only in virtue of extension, and it is therefore the same after the position of equilibrium is assumed as it was before; while in the first case the value of m at each point is not at once known, since in taking the position of equilibrium (*to which our equation of Virtual Work always refers*) extension has taken place at each point. In this case m at each point depends on T according to some law which can be known only by experiment—e.g. Hooke's Law,

$$m=\frac{m_0}{1+\frac{T}{\lambda}},$$

as in Art. 197.

The equations of the extensible and of the inextensible system are therefore the same only *in form*, since the above constitutes a vital distinction between them.

286.] Property of Minimum. *If a uniform inextensible string, in equilibrium under the action of a given conservative system of forces, joins two fixed points, A and B, the variation of the integral*

$$\int T ds$$

will be zero when we pass from the curve of the string to any indefinitely close curve which passes through A and B.

Let us calculate the variation of this integral.

$$\delta \int T ds = \int (\delta T . ds + T \delta ds)$$
$$= \int \left\{ \delta T . ds + T \left(\frac{dx}{ds} d\delta x + \frac{dy}{ds} d\delta y + \frac{dz}{ds} d\delta z \right) \right\}.$$

Now, from (6) of Art. 184, taking $k\sigma$, or m, as unity,

$$\delta T = -\delta V = -(X\delta x + Y\delta y + Z\delta z).$$

Hence by integration by parts (as in Art. 284), we have

$$\delta \int T ds = T_1 \left(\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z \right)_1 - T_0 \left(\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z \right)_0$$
$$- \int \left\{ \left[X + \frac{d}{ds}\left(T\frac{dx}{ds}\right) \right] \delta x + \left[Y + \frac{d}{ds}\left(T\frac{dy}{ds}\right) \right] \delta y \right.$$
$$\left. + \left[Z + \frac{d}{ds}\left(T\frac{dz}{ds}\right) \right] \delta z \right\} ds.$$

Now the right-hand side of this equation is zero, since, the extreme points of the curve being fixed, the coefficients of T_0 and T_1 both vanish, and the coefficients of δx, δy, δz under the sign of integration vanish by the general equations of Art. 284, the mass of a unit length of the string being here taken as unity. Hence the proposition.

This theorem leads to a remarkable property of the common catenary. *Of all curves of the same length joining two given points in a vertical plane, the common catenary is that whose centre of gravity is lowest.* For if $\bar{y}$ is the depth of the centre of gravity of this curve, whose length is L, we have

$$\bar{y} = \frac{\int y ds}{L}.$$

But (Art. 184) $T = mgy$; therefore $\bar{y} = \dfrac{\int T ds}{mgL}$; therefore, by the theorem of this article, we have

$$\delta \bar{y} = 0.$$

That $\bar{y}$ is in this case a minimum in the true sense of the word does not, of course, appear from this; the proof that it is so depends on the criterion for maxima and minima furnished by the Calculus of Variations, for which see Jellett's *Calculus of Variations*, p. 80. It is there proved, that when the variation of any integral of the form $\int_{x_0}^{x_1} U dx$ vanishes (the limits being fixed) the value will be, in general, an algebraic maximum or mimimum according as $\frac{d^2 U}{dp_n^2}$ is continually $-$ or continually $+$ between the limits of integration, $\frac{d^n y}{dx^n}$ being denoted by p_n, and U being any function of $x, y, p_1, p_2, \ldots p_n$. In the present case $U \equiv y ds = y\sqrt{1+p_1^2}\,dx$, a change of the independent variable from s to x being necessary since it is the limits of x that are assigned. The application of the criterion is then obvious.

287.] **Observations on the Method of Lagrange.** The application of the method of Lagrange is attended by a risk of error, which must be guarded against. In applying the equation of Virtual Work to any continuous material system—e. g. a string, a membrane, a fluid—we imagine every point to receive a small displacement from the position which it occupies in the equilibrium configuration of the system. These displacements we have expressed by increments $(\delta x, \delta y, \delta z)$, or (u, v, w) of the co-ordinates of the point; and, according to the nature of the system, there will be various relations between the u, v, w belonging to each point. Thus in the case of an absolutely inextensible string, these quantities have to satisfy at each point the equation

$$\delta(ds) = 0, \text{ or } \frac{dx}{ds}\frac{du}{ds} + \frac{dy}{ds}\frac{dv}{ds} + \frac{dz}{ds}\frac{dw}{ds} = 0.$$

In an absolutely free and unconnected system of particles, they have to satisfy no condition whatever.

Suppose that in any case they have to satisfy the condition of rendering a certain element—e. g. a length, an area, a volume—invariable. Suppose that this element is a function

$$\phi(dx, dy, dz, d^2x, \ldots),$$

which we may briefly denote by ϕ. Then *Lagrange's method consists in reducing the problem to a case in which we may treat u, v, w at each point in the system as absolutely independent, so that*

(as in Art. 284, for example) *we may equate their coefficients separately to zero.* This is done by taking the variation, $\delta\phi$, of the function which is to remain unaltered in the imagined displacement, multiplying it by an undetermined multiplier, λ, and then adding it under the sign of integration to the variation of the Potential Work of the system; so that our equation, in which u, v, w (or δx, δy, δz) are all independent, is

$$-\int \delta \Pi \,.\, dm + \int \lambda \delta \phi \,.\, dm = 0. \tag{A}$$

Now let the case be different. Suppose that the condition $\phi =$ constant has not to be satisfied in the displaced configuration, but that the alteration of ϕ is accompanied by internal forces (or *stresses*) in the system. In this case Lagrange makes no change in his mode of procedure. True, we have no longer the equation $\delta\phi = 0$, but Lagrange, recognizing the fact that we have internal work, or work done by the stresses, due to the displacement which alters ϕ, *assumes that the amount of this internal work is fully represented by a term of the form*

$$\lambda \delta \phi,$$

so that our equation of virtual work is still of the form (A).

It is this last assumption which is so liable to mislead, and which is, in more instances than one, a cause of error in Lagrange's own investigations. As a marked instance in which Lagrange has fallen into an error of this kind, we may cite his discussion of the equilibrium of a perfectly flexible surface, which may be (1) perfectly inextensible, or (2) extensible, like a sheet of indiarubber (see the *Mécanique Analytique*, p. 140).

Taking case (1), if dS is the *area* of the superficial element at any point P of the surface, Lagrange assumes that the only equation which u, v, w have to satisfy is $\delta dS = 0$; in other words, that perfect inextensibility is fully provided for if every element of the *area* remains unaltered in the (imagined) displaced configuration. But it is clear that perfect inextensibility requires that there shall be no alteration in the length of any line on the surface connecting P with a neighbouring point; and this characteristic is, therefore, expressed by *three* equations between u, v, w instead of one—as in Example 1, § 74.

Again, when the condition of inextensibility is removed, and the surface is extensible, Lagrange assumes that the internal

work of deformation of the element dS is fully represented by $\lambda \delta dS$, i. e. that the work of deformation is simply proportional to the change in the *area* deformed—an assumption which is true for membranes of few known materials.

On the other hand, the similar treatment of a string, whether inextensible or extensible, is perfectly valid, because $\delta ds = 0$ is a perfect expression of inextensibility; and when the string is elastic, the internal work of deformation is perfectly expressed by a term of the form $\lambda \delta ds$.

CHAPTER XV.

EQUILIBRIUM OF STRINGS AND SPRINGS.

297.] **Tangential and Normal Resolutions.** We now propose to complete the discussion of the equilibrium of flexible strings by considering the case in which the external forces are not coplanar.

Reverting to Fig. 221 of Chapter XII, consider the equilibrium of the element PQ apart from the rest of the string. Then the external force per unit mass at P will be, as before, of the form $\phi\,(x, y, z)$, where (x, y, z) are the co-ordinates of P; and the external force exerted on PQ will be

$$\phi\,(x, y, z) \times k\sigma ds, \text{ or } k\sigma F ds,$$

where k and σ are the density and area of normal section at P.

Now, the element PQ is kept in equilibrium by three forces —namely, the tension (T) at P, the tension $(T+dT)$ at Q, and the external force $(k\sigma F ds)$, which acts at the middle point of PQ.

These three forces must be coplanar and meet in a point. Now, the two tensions act along two consecutive tangents to the string, and as the plane of two consecutive tangents to any curve in space is the *osculating plane*, we see that—

The resultant applied force at any point of a flexible string acts in the osculating plane of the string at the point.

If the string is stretched over any smooth surface by means of two forces applied at its extremities, the only applied force which is *continuously* distributed throughout the string is the reaction of the surface; and as this reaction is everywhere normal to the surface, we see that—

A string which is stretched along any smooth surface, and acted on by no external forces, except the reaction of the surface and two terminal tensions, has its osculating plane at every point normal to the surface.

The string in this case assumes the form of a shortest line, or *geodesic*, on the surface.

Let Pt be the tangent and Pn the normal at P; let $d\theta$ be the angle between the tangents at P and Q; and let ϕ be the angle between Fdm and Pt.

Then, resolving along Pt the forces acting on the element, we have
$$(T+dT)\cos d\theta + k\sigma F\cos\phi ds - T = 0;$$
but $\cos d\theta = 1$, if we neglect $(d\theta)^2$; therefore this equation gives
$$\frac{dT}{ds} + k\sigma F\cos\phi = 0, \tag{1}$$
which asserts that the rate of variation of the tension per unit of length along the string is numerically equal to the tangential component of the applied force per unit of length.

Again, resolving the forces along Pn, the normal, we have
$$(T+dT)\sin d\theta - k\sigma F\sin\phi ds = 0,$$
or since ρ, the radius of curvature at P, is equal to $\frac{ds}{d\theta}$,
$$\frac{T}{\rho} - k\sigma F\sin\phi = 0, \tag{2}$$
which asserts that the curvature of the string at any point is equal to the normal force per unit of length divided by the tension.

From (1) we have $T = C - \int k\sigma F\cos\phi ds,$

where C is an arbitrary constant. Now, $\cos\phi ds$ is the projection of ds on the direction of F. Denoting this projection by df,
$$T = C - \int k\sigma F df. \tag{3}$$
But $\int k\sigma F df$ is evidently the potential of the applied forces if they are a conservative system. Hence, if V and V_0 denote the potentials at two points in the string at which the tensions are T and T_0, we have
$$T = T_0 - (V - V_0), \tag{4}$$
or *the difference of the tensions at any two points is equal to the difference of the potentials*—a result which we shall find to be true also in the case in which the string rests on a smooth surface.

298.] **Equations of Equilibrium.** Let the force F acting on the unit mass at any point P whose co-ordinates are x, y, z be resolved into three components, X, Y, Z parallel to three fixed

rectangular axes. Then the components acting on the element PQ are $k\sigma X ds$, $k\sigma Y ds$, $k\sigma Z ds$. Also the components of the tension acting on the extremity P are

$$-T\frac{dx}{ds}, \quad -T\frac{dy}{ds}, \quad -T\frac{dz}{ds};$$

the components of this tension are affected with negative signs, since, when the element PQ is considered apart, the tension at P will be directed towards the left-hand side of Fig. 221, where the origin of co-ordinates is supposed to be.

These components of the tension will at any point be functions of the length of the arc measured from some origin point, A, of the string up to the point considered. Thus, if $AP = s$, we shall have

$$T\frac{dx}{ds} = f(s),$$

and the component of the tension at Q is therefore $f(s+ds)$, or

$$T\frac{dx}{ds} + \frac{d}{ds}\left(T\frac{dx}{ds}\right)\cdot ds + \frac{d^2}{ds^2}\left(T\frac{dx}{ds}\right)\cdot\frac{ds^2}{1\,.\,2} + \ldots.$$

Hence, for the equilibrium of PQ, resolving forces parallel to the axis of x, we have

$$T\frac{dx}{ds} + \frac{d}{ds}\left(T\frac{dx}{ds}\right)\cdot ds + \frac{d^2}{ds^2}\left(T\frac{dx}{ds}\right)\cdot\frac{ds^2}{1\,.\,2} + \ldots$$
$$+ k\sigma X ds - T\frac{dx}{ds} = 0,$$

or, rejecting the terms which cancel, dividing out by ds, and diminishing ds indefinitely, and denoting $k\sigma$ by m, the mass per unit length,

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + mX = 0. \tag{1}$$

Similarly,

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) + mY = 0, \tag{2}$$

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right) + mZ = 0. \tag{3}$$

Performing the differentiations, we obtain

$$T\frac{d^2x}{ds^2} + \frac{dT}{ds}\frac{dx}{ds} + mX = 0, \tag{4}$$

$$T\frac{d^2y}{ds^2} + \frac{dT}{ds}\frac{dy}{ds} + mY = 0, \tag{5}$$

$$T\frac{d^2z}{ds^2} + \frac{dT}{ds}\frac{dz}{ds} + mZ = 0. \tag{6}$$

For the future we shall systematically use (α, β, γ) for the direction-angles of the tangent at any point P of a curve, the positive sense of this tangent being that in which the arc, s, measured up to P from some origin point on the curve, receives a positive increase.

Also by (ξ, η, ζ) we shall denote the direction-angles of the radius of absolute curvature at P, *taken in positive sense from P towards the centre of curvature.*

We may suppose these angles to be measured from lines drawn at P parallel to the positive directions of the axes of co-ordinates.

Equations (4), (5), (6) may then be written

$$\frac{T}{\rho}\cos\xi + \frac{dT}{ds}\cos\alpha + mX = 0, \tag{7}$$

$$\frac{T}{\rho}\cos\eta + \frac{dT}{ds}\cos\beta + mY = 0, \tag{8}$$

$$\frac{T}{\rho}\cos\zeta + \frac{dT}{ds}\cos\gamma + mZ = 0. \tag{9}$$

Multiplying these by $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and adding, we have

$$\frac{dT}{ds} + mS = 0, \tag{10}$$

where $S =$ the component of the forces along the positive sense of the tangent. This equation gives

$$T = C - \int S ds = C - \int m\,(Xdx + Ydy + Zdz),$$

which is obviously the same as (3) of last Article.

Again, eliminating T and $\frac{dT}{ds}$ from (7), (8), (9), we have

$$\begin{vmatrix} \cos\xi, & \cos\alpha, & X \\ \cos\eta, & \cos\beta, & Y \\ \cos\zeta, & \cos\gamma, & Z \end{vmatrix} = 0,$$

or
$$X\cos\theta + Y\cos\phi + Z\cos\psi = 0, \tag{11}$$

where (θ, ϕ, ψ) are the direction-angles of the normal to the osculating plane. This equation asserts—what is evident from first principles—that the resultant external force at any point lies in the osculating plane.

Another form of the value of T is obtained by integrating (1), (2), (3) separately, and squaring and adding. Thus

$$T^2 = (A - \int mXds)^2 + (B - \int mYds)^2 + (C - \int mZds)^2, \tag{12}$$

A, B, C being constants which must be determined after each integration by knowing the values of $T\frac{dx}{ds}$, ... at the point from which s is measured.

Again, by multiplying (7), (8), (9) by $\cos\xi$, $\cos\eta$, $\cos\zeta$ and adding,

$$\frac{T}{\rho} + mP = 0, \tag{13}$$

where P is the component force along the radius of curvature in the positive sense (i. e. towards the centre of curvature).

The equations of the curve formed by the string are obtained from (1), (2), (3) thus, by elimination of T,

$$\frac{A - \int mXds}{\frac{dx}{ds}} = \frac{B - \int mYds}{\frac{dy}{ds}} = \frac{C - \int mZds}{\frac{dz}{ds}}. \tag{14}$$

From (10) it follows that *if at no point of the string is there any component force along the tangent, the tension will be constant throughout.*

299.] **String on a Smooth Surface.** Now suppose that the string, while acted upon continuously by any forces, is placed on a smooth surface, which produces at each point a normal reaction, equal to Rds on the element of length ds at the point, P.

We shall denote by (l, m, n) the direction-angles of the normal to the surface in the sense in which R acts along the normal. Then we have simply to add the components $R\cos l$, $R\cos m$, $R\cos n$ to X, Y, and Z respectively in equations (1), (2), (3), or (7), (8), (9) of last Article, so that our equations are now

$$\frac{T}{\rho}\cos\xi + \frac{dT}{ds}\cos\alpha + mX + R\cos l = 0, \tag{1}$$

$$\frac{T}{\rho}\cos\eta + \frac{dT}{ds}\cos\beta + mY + R\cos m = 0, \tag{2}$$

$$\frac{T}{\rho}\cos\zeta + \frac{dT}{ds}\cos\gamma + mZ + R\cos n = 0. \tag{3}$$

If ω is the angle between the radius of curvature and the *inward* drawn normal to the surface at P (i. e. the normal drawn in the *sense opposite to that of R*), we have by multiplying these by $\cos l$, $\cos m$, $\cos n$, and adding,

$$R + mN - \frac{T}{\rho}\cos\omega = 0, \tag{4}$$

N being the normal force per unit mass in the sense of R.

By multiplying by $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and adding, we have

$$\frac{dT}{ds}+mS=0, \tag{5}$$

just as in the case of a free string.

When the applied forces have a potential, V, the integral of this equation, as in Art. 195, is

$$T=T_0-(V-V_0). \tag{6}$$

In the particular case in which the string rests on any smooth surface under the influence of gravity, this equation gives

$$T=T_0-mg(y-y_0), \tag{7}$$

the axis of y being a vertical line. From this it follows that all points at which the tension is the same lie in the same horizontal plane.

The curve of equilibrium of the string is found by eliminating T and R from the equations (1), (2), (3). Thus, if we eliminate first $\frac{dT}{ds}$ and R, we have

$$\begin{vmatrix} T\frac{d^2x}{ds^2}+mX, & \frac{dx}{ds}, & \frac{du}{dx} \\ T\frac{d^2y}{ds^2}+mY, & \frac{dy}{ds}, & \frac{du}{dy} \\ T\frac{d^2z}{ds^2}+mZ, & \frac{dz}{ds}, & \frac{du}{dz} \end{vmatrix}=0, \tag{8}$$

in which $u=0$ is the equation of the given surface, so that $\cos l:\cos m:\cos n=\frac{du}{dx}:\frac{du}{dy}:\frac{du}{dz}$. The value of T derived by integrating (5) must be substituted in (8), and we then get a differential equation which, with $u=0$, determines the curve.

300.] **String on a Rough Surface.** If a string, acted on by no forces, is stretched over a rough surface it need not, as in the case of a smooth surface, assume the form of a geodesic or shortest line. One simple case in which it will be a geodesic is that in which it is about to slip on the surface at every point in the direction of the tangent to the string at this point.

Geodesic. Consider the equilibrium of an element, PQ, of the string, whose length is ds, and suppose that it is about to slip in the direction QP. The element is acted upon by three forces—namely, a tension T, at P, a tension $T+dT$, at Q, and the total

resistance of the rough surface, which must pass through the intersection of the tangents at P and Q.

It is evident that we may consider this total resistance as acting at P, ultimately, since it is of the form $R_1\,ds$, R_1 being a finite quantity, and if it be assumed to act at any point between P and Q, its components in any directions will differ from those of the total resistance supposed to act at P by infinitesimals of the order of $(ds)^2$. Resolve the total resistance at P into a normal force, Rds, and a force in the tangent plane, μRds, μ being the coefficient of friction between the string and the surface.

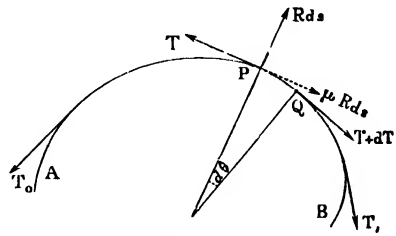

Fig. 265.

Now the component μRds must act along the tangent at P, since, by hypothesis, slipping is about to take place along this tangent. Hence the three forces T, $T+dT$, and μRds being all in the osculating plane of the curve at P, the remaining force, Rds, must also lie in this plane; that is, the osculating plane at every point of the curve contains the normal to the surface. Hence the string assumes the form of a geodesic.

Denoting the angle between the tangents at P and Q by $d\theta$, we have, by resolving along the tangent at P,

$$dT+\mu Rds = 0. \tag{1}$$

Again, resolving along the normal at P,

$$Td\theta - Rds = 0. \tag{2}$$

From (1) and (2) we have

$$\frac{dT}{T}+\mu d\theta = 0, \qquad \therefore \quad T = Ce^{-\mu\theta},$$

C being the constant of integration, and θ the sum of the *angles of contingence*, or angles between successive tangents to the string from any chosen point, A, to the point P. Let T_0 be the tension at A. Then $T = T_0$ when $\theta = 0$; therefore

$$T = T_0 e^{-\mu\theta}. \tag{3}$$

General Case. Suppose now that the string is acted upon by any forces, and that F is the force of friction per unit length at any point P, the direction of this force being in the tangent

plane, but otherwise unknown. Let its direction-angles be $(\alpha', \beta', \gamma')$. Then with the same notation as before,

$$\frac{T}{\rho}\cos\xi + \frac{dT}{ds}\cos\alpha + mX + R\cos l + F\cos\alpha' = 0, \tag{4}$$

$$\frac{T}{\rho}\cos\eta + \frac{dT}{ds}\cos\beta + mY + R\cos m + F\cos\beta' = 0, \tag{5}$$

$$\frac{T}{\rho}\cos\zeta + \frac{dT}{ds}\cos\gamma + mZ + R\cos n + F\cos\gamma' = 0. \tag{6}$$

Intrinsic equations, completely equivalent to the above, can be obtained by taking the axes of z, y, and x, respectively, parallel to the normal (direction of R), the tangent (direction of T), and a line drawn perpendicular to both, so that

$$\alpha = \gamma = \frac{\pi}{2}, \quad \beta = 0; \quad l = m = \frac{\pi}{2}, \quad n = 0;$$

$$\xi = \frac{\pi}{2} - \omega, \quad \eta = \frac{\pi}{2}, \quad \zeta = \pi - \omega.$$

If Q denotes the component force per unit mass at P along the new axis of x, and θ is the angle which the direction of F makes with the tangent, these equations become

$$\frac{T}{\rho}\sin\omega + mQ + F\sin\theta = 0, \tag{7}$$

$$\frac{dT}{ds} + mS + F\cos\theta = 0, \tag{8}$$

$$-\frac{T}{\rho}\cos\omega + mN + R = 0, \tag{9}$$

the last of which, therefore, holds both for a rough and for a smooth surface.

Consider the particular case in which there is no continuously applied external force, i.e. let $N = S = Q = 0$, and suppose that slipping is about to take place at a point. Then at this point $F = \mu R$, and we have

$$\frac{dT}{ds} = \frac{T}{\rho}\sqrt{\mu^2\cos^2\omega - \sin^2\omega}. \tag{10}$$

At a point, therefore, at which the osculating plane is inclined at the angle of friction to the normal to the surface, the tension is a maximum or minimum; and if slipping is about to take place at all points, the tension will be constant throughout if the osculating plane of the curve in which the

string is placed makes throughout the angle of friction with the normal.

If the osculating plane is everywhere normal to the surface, $\omega = 0$, and therefore $\sin\theta = 0$, i. e. the force of friction acts along the tangent—as is evident from the fact that of the four forces, T, $T + \frac{dT}{ds}ds$, R, and F, which keep an element in equilibrium, the first three are then coplanar, so that F must lie in the tangent.

Example.

A string whose weight may be neglected is placed along a circular section of a rough right cone and is pulled at its extremities by two given forces, P and Q; find the relation between these forces when the whole string is about to slip, and the direction of slipping at each point.

Ans. If α = semivertical angle of cone, μ = coefficient of friction, l = length of string, r = the radius of the circle, and if P is about to overcome Q,

$$Q = P\, e^{-\frac{l}{r}\sqrt{\mu^2\cos^2\alpha - \sin^2\alpha}},$$

and the direction of slipping makes at each point with the tangent the angle whose cotangent is $\sqrt{\mu^2\cot^2\alpha - 1}$.

301.] **Equilibrium of an Extensible String.** With the same notation as that employed in Art. 196, the equations of equilibrium of a flexible extensible string in the general case will be

$$m\,ds = m_0\,ds_0, \tag{1}$$

$$ds = \left(1 + \frac{T}{\lambda}\right) ds_0, \tag{2}$$

$$\left.\begin{aligned} \frac{d}{ds}\left(T\frac{dx}{ds}\right) + mX = 0,\\ \frac{d}{ds}\left(T\frac{dy}{ds}\right) + mY = 0,\\ \frac{d}{ds}\left(T\frac{dz}{ds}\right) + mZ = 0. \end{aligned}\right\} \tag{3}$$

$$ds = \sqrt{dx^2 + dy^2 + dz^2}, \tag{4}$$

$$m_0 = f(s_0). \tag{5}$$

In general, then, the two equations of the curve of equilibrium are found by eliminating m, m_0, s, s_0, T from these seven equations.

As before, we take only two cases, viz. that in which m_0 is constant, and that in which X, Y, Z are constant.

Firstly, consider m_0 constant.

Then, multiplying the left-hand sides of (3) by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, and adding, we get

$$\frac{dT}{ds} + m\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) = 0 ; \tag{6}$$

while from (1) and (2) we have $m = \dfrac{m_0}{1+\frac{T}{\lambda}}$; so that (6) gives

$$\left(1 + \frac{T}{\lambda}\right)dT + m_0\,(Xdx + Ydy + Zdz) = 0. \tag{7}$$

Integrating this and putting V for the potential of the external forces, per mass m_0 at the point (x, y, z), viz.

$$m_0 \int (Xdx + Ydy + Zdz),$$

we have
$$\frac{\lambda}{2}\left(1 + \frac{T}{\lambda}\right)^2 = A - V \tag{8}$$

where A is a constant.

Hence by (2)
$$\frac{ds}{\sqrt{A - V}} = \sqrt{\frac{2}{\lambda}}\,ds_0, \tag{9}$$

which gives s in terms of s_0, and therefore the extension.

If V and V' are the potentials at two points at which the tensions are T and T', respectively,

$$(T - T')\left(1 + \frac{T + T'}{2\lambda}\right) = V' - V. \tag{10}$$

The equations of the curve of equilibrium are obtained by substituting the value of T given by (8) in any two of the equations

$$\left(1 + \frac{T}{\lambda}\right)\frac{d}{ds}\left(T\frac{dx}{ds}\right) + m_0 X = 0,$$

$$\left(1 + \frac{T}{\lambda}\right)\frac{d}{ds}\left(T\frac{dy}{ds}\right) + m_0 Y = 0,$$

$$\left(1 + \frac{T}{\lambda}\right)\frac{d}{ds}\left(T\frac{dz}{ds}\right) + m_0 Z = 0.$$

Secondly, suppose the external forces X, Y, Z to be constant.

Then, integrating equations (3), we have

$$\left.\begin{aligned} T\frac{dx}{ds} &= A - X\int m_0 ds_0, \\ T\frac{dy}{ds} &= B - Y\int m_0 ds_0, \\ T\frac{dz}{ds} &= C - Z\int m_0 ds_0, \end{aligned}\right\} \tag{11}$$

A, B, C being constants. Squaring and adding these,

$$T^2 = (A - X\int m_0 ds_0)^2 + (B - Y\int m_0 ds_0)^2 + (C - Z\int m_0 ds_0)^2, \tag{12}$$

which gives T in terms of s_0. Suppose

$$T = \phi(s_0). \tag{13}$$

Hence from (2) $$s = \int \left\{1 + \frac{\phi(s_0)}{\lambda}\right\} ds_0,$$

from which the extension is known.

The equations of the curve are obtained from equations (11) by substituting for ds in terms of ds_0. Thus,

$$dx = (A - X\int m_0 ds_0)\left\{\frac{1}{\lambda} + \frac{1}{\phi(s_0)}\right\} ds_0,$$

$$dy = (B - Y\int m_0 ds_0)\left\{\frac{1}{\lambda} + \frac{1}{\phi(s_0)}\right\} ds_0,$$

$$dz = (C - Z\int m_0 ds_0)\left\{\frac{1}{\lambda} + \frac{1}{\phi(s_0)}\right\} ds_0.$$

Integrating these and eliminating s_0, we obtain the two equations of the curve of equilibrium.

302.] **Equilibrium of a Plane Elastic Rod.** The equilibrium of a string has been investigated, in Art. 297, on the supposition that if we take a normal section of it at any point, P, the action exerted on the portion PB by the remaining portion PA consists simply of a force directed along the tangent. The rod differs from the string in this—that

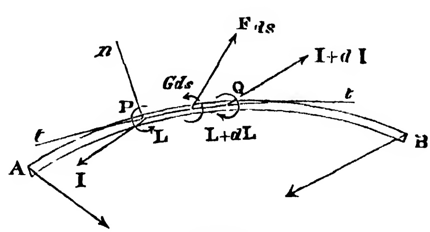

Fig. 266.

the internal action exerted on any normal section is much more

complicated, being equivalent to a force, I, acting at some point of the section, oblique to the tangent, together with a couple L. In the case, now before us, of a rod lying wholly in one plane and acted upon by external forces and couples, also confined to this plane, the axis of the couple, L, will at every point be perpendicular to the plane of the rod. Indeed, the remarks in Art. 103 on the nature of internal action, or stress, prepare us for seeing this.

In the above figure (Fig. 266), consider the nature of the action exerted over the normal section at P on the part PB by the part PA. Near the upper, or convex, side the bending has the effect of making the part PA try to tear away from PB, so that, on the whole, there will be in this neighbourhood forces on PB directed towards the *left*; while near the lower, or concave, side of the rod at P, the bending causes the portion PA to push into PB, and consequently the particles of PB in this neighbourhood will experience forces directed towards the *right* of the figure.

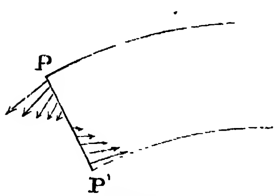

Fig. 267.

This state of stress is roughly represented in Fig. 267. If the arrows in it represent the forces experienced by the individual molecules of the portion PB, it is clear how such forces might reduce to a single force acting below P', perhaps a long way off from the rod; and how this single force, again, could (Art. 79) be replaced by the force I (Fig. 266) acting at an arbitrary point of the section, together with the counter-clockwise couple L.

A remark may be made with reference to the system of stress (Fig. 267) on the section of the rod. It is this—that even though the normal section may be extremely small, as in the case of a very narrow wire, the forces experienced by the successive molecules lying in the section vary in both magnitude and sense with enormous rapidity. On an *infinitely* small area of this normal section—such as the surface of a single atom—the stress action consists necessarily of a force simply—without any couple; but the normal section of even a very thin wire contains an infinitely great number of such infinitely small surfaces, and therefore furnishes abundant possibility for the enormously rapid change in the magnitude and sense of the separate internal forces,

and hence for that force and couple to which, if the wire be very stiff, these individual forces must reduce.

The rod, whose equilibrium we are considering, is supposed, for generality, to be acted upon continuously throughout its length by an applied force, whose amount per unit length at any point is F, together with an applied couple, whose amount per unit length is G. A magnetized spring acted upon, in addition to any other forces, by the earth's magnetic attraction, gives an instance of continuously distributed external couple.

303.] **Conditions of the Extremities.** Our figure represents the rod as kept in equilibrium by continuously distributed force (Fds) and couple, together with two terminal forces at A and B. These terminal forces may be produced either by direct pulls or by fixing smooth pins through the extremities, since (Art. 103) the pressures all round the surface of a smooth cylindrical axis are equivalent to a single force acting through the centre of the axis.

Another, and essentially different, state of affairs at the ends is produced by fixing not only the end itself but also the tangent at it. In this case we shall speak of the end as *tangentially fixed.* It is clear that this mode of fixture could not be produced by the application of a *single force* at the end so fixed; it would require the application of a *force and a couple* to the end. In the case of coplanar forces this force and couple are equivalent to a single force acting at a distance from the end (Art. 79).

Pivoting at an extremity is, then, productive of a single force acting at the extremity; and *tangential fixture* is productive of a force and a couple acting at it.

304.] **Equations of Equilibrium.** We now proceed to obtain the equations connecting the stress with the external forces and couples, exactly as in the case of a string. Consider the separate equilibrium of the element PQ. The stress which it experiences at P has been already described; over the normal section at Q, the stress will consist of a slightly different force, $I+dI$, and a slightly different couple, $L+dL$; while the externally applied force is Fds, and the externally applied couple is Gds. (The force $I+dI$ and the couple $L+dL$ are exerted by the portion QB on QA, and are therefore in the senses represented in the figure.)

Suppose the arc s to be measured from A, so that $AP = s$;

let Sds and Nds be the components of Fds along the tangent tP and the normal Pn drawn towards the convex side of the curve; let θ be the angle which the tangent at P makes with some fixed line (axis of x) which we may, for definiteness, suppose drawn at the lower side of the figure, so that the radius of curvature, ρ, at P is $-\frac{ds}{d\theta}$. Also, let the internal force, I, be resolved into components, T, along the tangent, and U, along the normal. The second component is called the *shearing stress* at P; the first is, of course, the *tension* of the rod. Let the tension and shearing stress at Q be $T+dT$ and $U+dU$, respectively.

Then, for the equilibrium of PQ, resolving along the tangent, we have

$$-T+T+dT-(U+dU)\,d\theta+Sds=0,$$

observing that $d\theta$ is negative. Hence, proceeding to the limit,

$$\frac{dT}{ds}+\frac{U}{\rho}+S=0. \tag{1}$$

Similarly, by resolving along the normal,

$$U+dU-U+(T+dT)\,d\theta+Nds=0.$$

$$\therefore\quad \frac{dU}{ds}-\frac{T}{\rho}+N=0. \tag{2}$$

Finally, taking moments about an axis through P perpendicular to the plane of the figure, and observing that the moment of the external force would give a term of the order ds^2, we have

$$L-L-dL+(U+dU)\,ds+Gds=0,$$

$$\therefore\quad \frac{dL}{ds}-U-G=0. \tag{3}$$

From this last we see that when there is no *continuously* distributed external couple, *the shearing stress at any point is equal to the differential coefficient of the stress couple with respect to the arc.*

With regard to the sense of the stress couple L, observe, in general, that the couple exerted on any portion PA by the remaining portion PB is in the sense in which the tangent at P revolves as we move from P along PB; and in (3) the shearing stress exerted on PA by PB is measured along the normal drawn towards the *convex* side of the curve. If U is measured towards the centre of curvature we have simply to change its sign in the equations.

Sometimes it is of more advantage to obtain equations from the consideration of the equilibrium of a portion of finite length of the curve.

Thus, consider the equilibrium of the whole length AP. Take any two fixed axes of x and y; let Xds and Yds be the components of the external force, Fds, parallel to these axes; let α and β be the components of the force (arising from any such cause as fixture) at A, and let λ be the special couple (if any) applied at A. Then, by resolution, we have

$$T\cos\theta - U\sin\theta = \alpha - \int X ds, \tag{4}$$

$$T\sin\theta + U\cos\theta = \beta + \int Y ds, \tag{5}$$

which give T and U at once.

Also, by taking moments about P, and denoting the co-ordinates of A by a and b, while, to avoid confusion, we denote the co-ordinates of any point on the curve between A and P by ξ and η, we have

$$L = \lambda + \alpha(y-b) - \beta(x-a) + \int\{X(y-\eta) - Y(x-\xi) + G\} ds. \tag{6}$$

Or the value of L may often be better obtained from (3), U having been determined from (4) and (5).

305.] Particular Case of Plane Spring. Suppose that there are no forces or couples *continuously* distributed along the rod, or spring, but merely a force, H, at one end, B, the other being either simply or tangentially fixed; and suppose that before strain the spring had the form of any plane curve, of which the radius of curvature at any point was r. Considering the equilibrium of any portion PB, we see that stress at P (force and couple) must reduce to a force equal to the force H and directly opposed to it in its line of action.

Hence at all points the internal force I is constant in magnitude and direction, being equal to H.

Again, by moments about P, if we take the line of action of H as axis of x, we have

$$L = H.y. \tag{1}$$

Now the magnitude of L is assumed to be equal to the change in curvature at P produced by strain, multiplied by a certain constant, A, whose magnitude depends on the stiffness of the material of the spring; so that

$$L = A\left(\frac{1}{\rho} - \frac{1}{r}\right). \tag{2}$$

The constant A is called a *flexural rigidity*, and it is evidently of the nature of a force multiplied by the square of a line.

We may therefore put $\frac{A}{H} = a^2 =$ a constant, and then we have

$$\frac{1}{\rho} = \frac{1}{a^2}\left(y + \frac{a^2}{r}\right), \tag{3}$$

which is the equation determining the form of the curve.

If the spring when free from strain was straight, $\frac{1}{r}$ is zero, and the equation becomes

$$\frac{1}{\rho} = \frac{y}{a^2}. \tag{4}$$

If the rod was in the form of a circular arc when unstrained, (3) could be put into the form (4) by taking the axis of x parallel to the line of action of H at a distance $\frac{a^2}{r}$ from it.

The force, H, may be applied either directly to the end, B, of the rod itself or to a rigid arm attached to B.

The latter case is the same as if H were directly applied to B and accompanied by a couple whose moment is the moment of H about B—in fact, a rigid arm at the extremity of which H is applied may be regarded as a means of applying a force and a couple at the end, B, of the rod (see Art. 202).

306.] **Elastic Curves and Elliptic Functions.** Let ADB (Fig. 268) represent the rod, with two rigid arms, Aa, Bb,

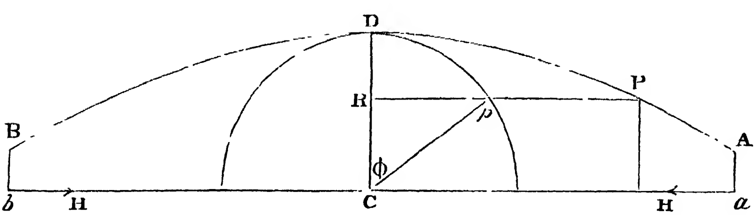

Fig. 268.

attached to its ends, two equal and directly opposed forces, H, being applied perpendicularly to these arms. We assume that the rod was straight when unstrained.

Taking the line ab as axis of x, the equation of the bent rod is

$$\rho y = a^2. \tag{1}$$

We shall express ρ in terms of the element of arc, ds, and the angle, $d\theta$, between the tangents at its extremities; and for definiteness we shall assume s to be measured from a point, D, at which the tangent is parallel to the line, ab, of action of the force H, and the angle θ made with the tangent at D by the tangent at any point, P, on the curve to be measured positively in the sense of clockwise rotation.

The curve of equilibrium may be concave at some points and convex at others to the line of action of the terminal forces; and if at any point it intersects this line, its curvature vanishes at the point. It may, again, never intersect this line at all.

If P and Q are any two very close points on the curve, the extremity of the curve, which we should reach by travelling from P to Q and then continuously along the curve, may be called *the extremity adjacent to* Q, while the other extremity may be called *the extremity adjacent to* P.

The terminal forces, H, may act along ab either towards each other, as represented in Fig. 268, or from each other, and the sense of the bending (or concavity) at any point will depend on the senses of the terminal forces. In every case, of course, the sense of the bending at any point P is such that the moment about P of the terminal force at either extremity is opposed by the stress couple exerted at P by the remaining portion of the rod; or, in other words, for all the figures which the curve can assume we have the following rule—*the sense in which the tangent revolves in passing from* P *to a consecutive point* Q *is the sense of the moment about* P *of the force at the end adjacent to* Q.

If in (1) we put $\rho = \frac{ds}{d\theta}$; and $\frac{dy}{ds} = -\sin\theta$, we have

$$a^2 \frac{d^2\theta}{ds^2} = -\sin\theta, \tag{2}$$

$$\therefore \quad a^2 \left(\frac{d\theta}{ds}\right)^2 = C + 2\cos\theta,$$

where C is a constant. Let D (Fig. 268) be a vertex, or point at which the tangent is parallel to ab, and let the ordinate $DC = h$; then from the last equation we have

$$y^2 = h^2 - 4\,a^2 \sin^2 \tfrac{1}{2}\,\theta. \tag{3}$$

Now different cases arise according as h is $< 2a$, $= 2a$, or $> 2a$.

CASE 1 ; $h < 2a$. Let $h = 2ak$, where k is a fraction. Then

$$y = 2a\sqrt{k^2 - \sin^2\tfrac{1}{2}\theta}. \tag{4}$$

Let $\sin\frac{1}{2}\theta = k\sin\phi$; then

$$y = h\cos\phi. \tag{5}$$

The angle ϕ can be easily exhibited: with C, the foot of the ordinate from D, as centre, describe a circle of radius h; from P, which is any point on the curve, draw PR parallel to ab, meeting the circle in p. Then the angle DCp is obviously ϕ.

To find the length of the arc DP, or s, we have $\dfrac{dy}{ds} = -\sin\theta$;

but from (5), $\dfrac{dy}{ds} = -h\sin\phi\dfrac{d\phi}{ds}$; hence

$$\frac{ds}{d\phi} = \frac{h\sin\phi}{\sin\theta}$$

$$= \frac{a}{\sqrt{1-k^2\sin^2\phi}},$$

$$\therefore \quad s = a\int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \tag{6}$$

$$= a\,.\,F(k,\phi), \tag{7}$$

according to the notation for an elliptic integral of the first kind.

To express the abscissa, CM, of P. If $CM = x$, we have

$$\frac{dx}{ds} = \cos\theta, \quad \therefore \quad \frac{dx}{d\phi} = h\sin\phi\cot\theta = a\frac{1-2k^2\sin^2\phi}{\Delta\phi},$$

where $\Delta\phi \equiv \sqrt{1-k^2\sin^2\phi}$. Hence

$$\frac{dx}{d\phi} = 2a\,.\,\Delta\phi - \frac{a}{\Delta\phi}, \tag{8}$$

$$\therefore \quad x = 2a\,.\,E(k,\phi) - a\,.\,F(k,\phi), \tag{9}$$

$$\therefore \quad x + s = 2a\,.\,E(k,\phi), \tag{10}$$

where, as usual, E denotes the elliptic integral of the second kind.

EXAMPLES.

1. If the ends A and B are fixed by smooth pins at a given distance apart, and the rod is placed in the form of n bays, or spans, between the pins, find the pressures exerted by the pins.

The pressures exerted by the pins will be directly opposed in the line AB. Let $AB = c$, and let $l =$ whole length of the rod. Then

the length of the arc in one span is obtained from (7) by putting $\phi = \frac{1}{2}\pi$. Hence

$$l = 2naF(k, \tfrac{1}{2}\pi). \tag{11}$$

Similarly, from (10), we have

$$c+l = 4naE(k, \tfrac{1}{2}\pi); \tag{12}$$

so that k is determined from the equation

$$2l \,.\, E(k, \tfrac{1}{2}\pi) = (c+l) \,.\, F(k, \tfrac{1}{2}\pi), \tag{13}$$

which is independent of the number of bays.

Now there is only one value of k which satisfies this equation, as we can see graphically thus. The values of k range from 0 to 1. Draw two rectangular axes, Ox and Oy, and let abscissae (along Ox) represent values of k while ordinates represent the corresponding values of E.

When $k = 0$, $E = \frac{1}{2}\pi$, and when $k = 1$, $E = 1$. Also, we easily find by differentiation that

$$\frac{dE}{dk} = \frac{E-F}{k}, \tag{14}$$

$$\frac{dF}{dk} = \frac{1}{k}\left(\frac{E}{k'^2} - F\right), \tag{15}$$

in which we use E and F, for shortness, instead of $E(k, \frac{1}{2}\pi)$ and $F(k, \frac{1}{2}\pi)$; and also k'^2 for $1-k^2$.

Measure off $OV = \frac{1}{2}\pi$ along Oy, and $OT = 1$ along Ox, and at T draw an ordinate $TR = 1$. Then the curve representing the values of E passes through V and R, touching TR at R, and touching at V a line parallel to Ox. (The value of $\frac{dE}{dk}$ at V assumes the form $\frac{0}{0}$, but it is easy to find, by the help of (15), that it is equal to zero.) This curve is continuously concave towards the axis of x.

Again, the curve whose ordinates represent the values of F passes through V, touching the previous curve at this point, its ordinate being thenceforth always $> OV$, until when $k = 1$, the ordinate $= \infty$. This curve, therefore, approaches the line TR asymptotically. An inspection of the figure shows that if the ordinates of these curves have a given ratio, there is only one value of k which will answer, since the second curve is continuously convex towards the axis of x, E being always $> k'^2 F$, except at the point V.

The value of k, then, must be found empirically from (13), and if the corresponding value of F is μ, equation (11) gives

$$H = 4n^2\mu^2 A/l^2 \tag{16}$$

for the pressure exerted by each pin.

Hence the pressure is proportional to the square of the number of bays.

If α is the angle made with the line AB by the tangent at either extremity of the rod, since, in general, $\sin\frac{1}{2}\theta = k\sin\phi$, we have

$$\sin\tfrac{1}{2}\alpha = k, \tag{17}$$

for any figure which crosses the line of force, since for all such curves $\phi = \frac{1}{2}\pi$ for the point of crossing.

In the present case, therefore, whatever be the number of bays, their terminal tangents are all equally inclined to the line AB.

The particular case in which the ends A and B are brought together deserves to be noticed. One form of equilibrium is, of course, that of a single loop starting from A and coming round to A again, there being two distinct tangents to the curve at A.

For this case put $c = 0$ in (13), and the value of k is obtained from the equation

$$2E(k, \tfrac{1}{2}\pi) = F(k, \tfrac{1}{2}\pi), \tag{18}$$

and the inclination of each tangent is given by (17).

Another form of equilibrium in this case is that of a figure of 8, the two tangents at the double point making with the axis of the curve the same angles as those just found for the case of a single loop.

2. Show that if a rod is slightly bent between its extremities, its figure is that of the curve of sines, $y = h\sin x/a$.

Case 2; $h = 2a$. In general, the radius of curvature at the vertex D is equal to a^2/h, so that when $h > a$, the curve on leaving D comes inside the circle Dp (Fig. 268). Such happens, then, in the present case; and we easily find from (7) and (10)

$$s = a\log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi), \tag{19}$$

$$x = 2a\sin\phi - a\log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi). \tag{20}$$

Also $\phi = \frac{1}{2}\theta$, and $y = 2a\cos\frac{1}{2}\theta$. On leaving the vertex, D, the value of x begins by increasing; but the logarithmic term in (20) must soon destroy the term $2a\sin\phi$, so that $x = 0$, or the curve cuts the axis of y at some such point as J (Fig. 269).

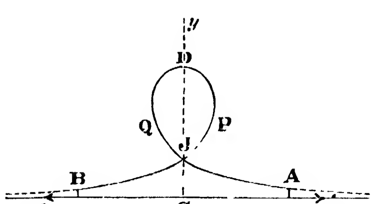

Fig. 269.

The course of the curve from D is $DPJB$; and there is obviously a similar and equal portion represented by $DQJA$; and the productions of the curve beyond the arms, at the extremities of which the forces H are applied, are asymptotic to the line of force.

Case 3; $h > 2a$. In this case put $2a = k.h$, where k is, of

course, < 1. Hence $y = h\sqrt{1-k^2\sin^2\frac{1}{2}\theta}$; and if we put $k\sin\frac{1}{2}\theta = \sin\phi$, we have $y = h\cos\phi$, so that we have the same geometrical representation of ϕ as before. But in this case the curve can never cross the line of force, since $\sin\frac{1}{2}\theta$ cannot be equal to $1/k$, and, consequently, y cannot vanish.

The following results are easily found:

$$s = ka \, . \, F(k, \tfrac{1}{2}\theta); \quad x = \frac{2a}{k} \, . \, E(k, \tfrac{1}{2}\theta) - \frac{2-k^2}{k} a \, . \, F(k, \tfrac{1}{2}\theta).$$

The form of the curve is that represented in Fig. 270, in which ab is the line of action of the terminal forces. This figure represents also the curve of what is called the *Hydrostatic Arch* (omitting, of course, the looped portions).

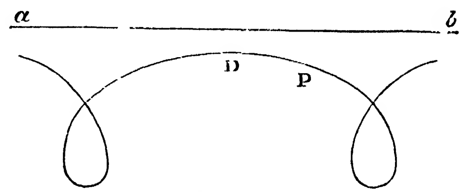

Fig. 270.

The idea in the construction of this arch is as follows: If a perfectly flexible string with fixed ends is in equilibrium under the action of continuously applied external force, which is everywhere normal to the string, the tension, T, is constant, and at each point it $= N\rho$, where N is the normal force per unit length and ρ the radius of curvature at the point (Art. 183). In such a system there is no shearing force and no stress couple at any point. Now, if we imagine the sense of every force to be reversed, i. e. let the normal force, N, be converted into pressure towards the centre of curvature, and the tension, T, to be converted into thrust, while the string becomes a body capable of resisting tangential thrust (i. e. a body of the nature of a wire), *no shear and no bending couple would be called into play in the new body*. But these are the objects to be desired in an arch, DP, supporting water, since with no shear or bending couple, there will be no tendency in the joints to separate. The stress in the arch will then consist of direct tangential thrust between stone and stone.

Now if P is any point on the arch, and ab the level of the superincumbent water, the pressure per unit area at P is wy, where w is the weight of the water per unit volume, and y the depth of P below ab. If, then, the arch is merely a rigidified

string devoid of shear and bending stress, $T' =$ constant, and $\frac{T}{\rho} = wy$, therefore $\rho y =$ constant, where T is the thrust in the arch per unit of breadth (i. e. per unit distance perpendicular to the plane of the figure).

Practically the elastic curve $\rho y = a^2$ can be constructed as an assemblage of small circular arcs thus. Take any axis of abscissae, ab (Fig. 268), and, starting with a point D, describe a small circular arc, DD', whose centre is on the ordinate DC. Let this centre be O, and let y' be the ordinate of D'. Then on the line $D'O$ take a point O' such that $O'D' = OD.\frac{y}{y'}$, where y is the ordinate, DC, of D, and with O' as centre draw the small circular arc $D'D''$; continue by a small circular arc from D'' to D''', and so on, and we get an approximate figure of an elastic curve.

A plane flexible string, every element of which is acted upon by a *normal* external force only, whose magnitude is proportional to the distance of the element from a fixed line in the plane of the string, assumes the form of one of the elastic curves, since by equation (2) of Art. 183, we have

$$\rho y = \text{constant}.$$

This would be the case of a flexible cylindrical sheet filled with water. A section of the surface perpendicular to the axis of the cylinder would—at least in places not near the ends of the cylinder—be a curve satisfying the equation $\rho y =$ constant, since each element is acted upon by a normal force (the water pressure) whose magnitude is proportional to the depth, y, of the element below the free surface of the water—the lateral tensions proceeding from the elements of the sheet outside the plane of the section contributing no component tension in the plane of the section.

For information as to twisted wires, Thomson and Tait, *Natural Philosophy*, vol. ii, may be consulted.

CHAPTER XVI.

THEORY OF ATTRACTION.

SECTION 1.—*Direct Calculation of Attraction.*

315.] **Newtonian Law of Attraction.** The motions of the planets and comets of the solar system can be explained completely on the hypothesis that each body of this system attracts every other body of the system with a force which in any position of the two bodies is directly proportional to the product of the masses of the bodies, and which in different positions is inversely proportional to the square of the distance between them. The fact that the positions which will be occupied by comets can be predicted with certainty, that the existence of Neptune was mathematically deduced from the assumption that certain disturbances in the motion of Uranus were due to the attraction of an unknown planet according to the above law, and several other facts of the same kind, all prove that the law holds with all the accuracy that human observation is capable of testing, so far as the action upon each other of large masses separated by distances which are great compared with their linear dimensions is concerned.

As to the cause, or mechanism, to which this attraction is due, nothing is known. Newton says in the Scholium at the end of Book III of *The Principia*, 'To us it is enough that gravity does really exist and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies and of our sea.' A little before this in the same Scholium he says, 'But hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypothesis (*hypotheses non fingo*).'

Although Newton framed no hypothesis on the mode by which gravitation is propagated through space, he mentions certain speculations which were current in his time, and which have been brought into great prominence in our days. Thus, at the end of section xi of Book I he says, 'I here use the word attraction in general for any endeavour, of whatever kind, made by bodies to approach each other; whether that endeavour arise from the action of the bodies themselves, as tending mutually to or agitating each other by spirits emitted; or whether it arises from the action of the æther or of the air, or of any medium whatsoever, whether corporeal or incorporeal, anyhow impelling bodies placed therein towards each other.'

By far the most promising step that has been taken towards a solution of this great difficulty is the discovery by Faraday that the attraction between two electrified bodies is influenced by the insulating medium in which they are placed, inasmuch as this discovery renders it highly improbable that any force produced by one body on another is a *direct action at a distance.* This discovery has been worked by Clerk Maxwell into a consistent mathematical theory of the mechanism by which magnetic and electromagnetic actions are propagated by a rare medium filling space.

Newton does not, however, confine the law of attraction according to the inverse square of distance to large masses like the planets; for he investigates the attraction of a solid on a particle, even when the particle is within the matter forming the body, on the supposition that this particle is attracted by *every* elementary particle of the body—however close to the attracted particle—with a force expressed by this law.

The assumption that *every indefinitely small particle of matter attracts every other particle with a force which acts in the right line joining the particles and whose magnitude is directly proportional to the product of the quantities of matter in the particles and inversely proportional to the square of the distance between them* is the formula of what is called the *Law of Universal Gravitation.*

The terms of the enunciation render it clear that *the linear dimensions of the particles must be infinitely small compared with the distance between them*—otherwise, indeed, the term 'distance between them' would be unmeaning. We shall soon prove,

however, that if the particles are homogeneous and spherical, this limitation may be removed, and the 'distance between them' is the distance between their centres.

But it is just at this point—i.e. when dealing with forces exerted on each other by indefinitely close molecules—that our ignorance of the cause or mechanism of this attraction introduces a most unsatisfactory dualism—or rather *multiplicity* of laws—into physical science.* For we are often presented with *repulsions* instead of attractions, and the phenomena of Elasticity and Capillarity have hitherto compelled physicists to assume other laws of force between molecules than the Newtonian law of the inverse square of distance, or the *law of nature*, as it is often called.

Electrical and magnetic attractions and repulsions are proved by experiment to obey this law, and therefore the theory of attraction is almost wholly a discussion of its consequences.

The quantitative expression of the Newtonian law is as follows. Suppose two very small particles whose masses are m grammes and m' grammes to be placed at a distance of r centimetres apart—this distance being, as before said, very great compared with the linear dimensions of either particle; then each will attract the other with a force equal to

$$\gamma \frac{mm'}{r^2} \text{ dynes,} \tag{α}$$

in which expression γ is an absolute constant, i.e. one whose magnitude is independent of the magnitudes of the masses and their distance.

We shall subsequently calculate the value of γ, which is called the *absolute constant of gravitation.* With the units of mass and distance chosen as above, γ is evidently *the number of dynes in the force with which a mass of one gramme condensed into an infinitely small volume attracts an equal mass similarly condensed*

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* For example, Clerk Maxwell, in his article on Capillary Action (*Encyclop. Brit.*) says: 'The forces which are concerned in these phenomena are those which act between neighbouring parts of the same substance, and which are called forces of cohesion, and those which act between portions of matter of different kinds, which are called forces of adhesion. These forces are quite insensible between two portions of matter separated by any distance which we can directly measure. It is only when the distance becomes exceedingly small that these forces become perceptible.'

Clearly science still needs a vigorous application of Occam's Razor.

at a distance of one centimetre; or, as we shall see, the force of attraction between two homogeneous spherical grammes with a distance of one centimetre between their centres.

316.] **Conical Angles.** Let $ABCDE$ (Fig. 273) be any closed curve, plane or tortuous, and O any point. If from O lines OA, OB, &c., are drawn to every point on the curve, we obtain a cone. If round O a sphere of 1 centimetre radius is described, the rays OA, OB, &c., constituting the cone will intersect the spherical surface in a curve $abcde$; and the number of square centimetres in the area of the spherical surface contained within this curve is called the *solid angle* subtended at O by the given curve $ABCDE$. Instead of this term (which is in no way expressive) we shall use the term *Conical Angle.* If the sphere described round O has a radius of 1 mile instead of 1 cm.. the number of square miles of the spherical surface enclosed by $abcde$ will be the conical angle, and this number will be the same as that of square centimetres on a sphere of radius 1 cm. Generally, if a sphere of any radius, r, be described round O, and the curve $ABCDE$ conically projected, as above, on its surface, the ratio of the area of $abcde$ to the square of the radius r is the measure of the conical angle subtended at O by the given curve—just as the *plane* angle subtended at O by any two points, P, Q, is the ratio of the length of the arc of any circle, with O as centre in the plane POQ, intercepted by the rays OP and OQ, to the length of the radius.

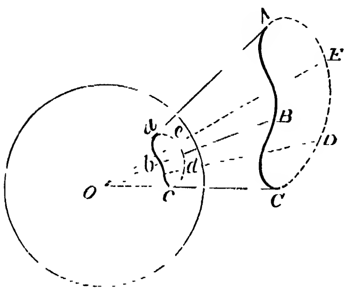

Fig. 273.

A conical angle is thus a mere *number*, like the circular measure of a plane angle.

The sum of all the conical angles round any point is 4π, because it is the whole area, in square centimetres, of a sphere of 1 cm. radius described round the point.

The conical angle subtended by any closed plane curve at any point which is in the plane of the curve and inside its area is 2π, since the rays OA, OB, &c., from O to the different points

on the curve intersect a spherical surface described round O as centre in a great circle of the sphere.

Let any closed *surface* be broken up into an indefinitely great number of small elements of area; then the sum of all the conical angles subtended by the contours of these elements at any point, O, inside the given closed surface is obviously 4π.

If O is *anywhere* on the surface itself, the sum of all the conical angles subtended at O by the elements of area of the surface is 2π, since the slender cones revolving round O lie all on one side of the tangent plane at O, and they will cut off only the area of half the sphere described round O.

If O is anywhere *outside* the given closed surface, the sum of all the conical angles subtended at O by the elements of area on the surface is *zero*. This case requires a little explanation.

Let any line drawn through O meet the given closed surface in points P_1, P_2, P_3, P_4 (Fig. 274), of which there must be an even number; and let a very slender cone of rays drawn through O intersect the surface in the small elements of area represented at these points. Then although *numerically*

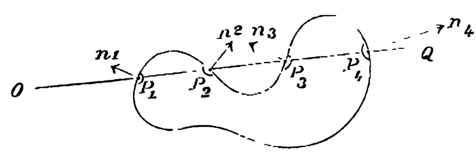

Fig. 274.

the conical angles subtended at O by these elements of area are all the same, distinctions of *sign* must be made between them. These distinctions can easily be made thus. At P_1 it is the *outside* of the surface that is turned towards O; at P_2 it is the *inside*; at P_3 the outside; and at P_4 the inside. Hence if $d\omega$ is the magnitude of the conical angle subtended at O by these elements, we may agree to make it *plus* for the inside aspects, P_2 and P_4, and *minus* for the outside aspects, P_1 and P_3; so that the sum of the conical angles subtended at O by these four elements of the given surface is zero.

For the purpose of projecting any element of area—as that at P_1—on any plane, we may adopt the convenient and consistent plan of drawing at the point the normal $P_1 n_1$, *outwards* from the surface proportional in length to the element of area, marking its extremity with an arrowhead, thus treating it as

a *directed* magnitude, like a force, and taking its *component* along the normal to the plane as representing in magnitude and sign the projection of the element of area at P_1 along the plane in question.

Thus, the conical angle subtended at O by the element, dS_1, of area at P_1 is represented by the projection of Pn_1 along the line OQ which is the normal to the surface of the sphere of projection; this gives the conical angle $= dS_1 \times \cos n_1 P_1 Q$, which is negative. Similarly for the other points, P_2, P_3, P_4.

If dS is any element of area of a surface at a point P, and $d\omega$ is the conical angle subtended at any point O by this element, while ψ is the angle between OP and the outward-drawn normal at P, we have

$$dS = \frac{OP^2}{\cos\psi} \cdot d\omega. \qquad (\alpha)$$

For, if a sphere is described through P having O for centre, the cone of rays which intercepts the area $d\omega$ square centimetres on the sphere of 1 cm. radius will intercept on this sphere an area of $OP^2 . d\omega$ square centimetres (if OP is measured in centimetres); and since this is the projection of dS on the surface of the sphere, we have the result (α).

The locus of the point O at which a given closed curve, or *circuit*, subtends a constant conical angle is a surface which contains the given curve as an edge—just as *in plano* the locus of a point O at which two fixed points, A, B, subtend a constant angle is a curve (circle) passing through A and B. The constant angle belonging to any one of a series of circles passing through A and B may be found by joining any point on the circle to A and B; but if the point selected on the circle is either A or B itself, an indeterminateness naturally arises, since the line joining B to itself is indeterminate. However, for any one circle if we take a point on the curve infinitely close to B, the direction of the line joining it to B is the tangent to the circle at B; so that the angle pertaining to that circle is the angle between AB and the tangent to the circle at B.

Similarly when the point O is taken on the given circuit, the conical angle subtended at it by the curve is naturally indeterminate; and to determine the angle pertaining to any one surface of the series of surfaces of constant conical angle having the given circuit for an edge, we must take a point, O', infinitely close to O *in the tangent plane to the particular surface.* The rays joining O' to the various points on the neighbouring part of the circuit form a semicircular fan of rays in the tangent plane, and they will intersect the sphere of unit radius described round O as centre in a semicircle; thus the

projection of the given circuit (which projection answers to the curve *abcde* in Fig. 273) on the unit sphere at O consists of a great semicircle and some irregular curve, U (suppose), completing this semicircle into a closed curve on the sphere; and the area of the sphere inside this closed curve is the conical angle belonging to the selected surface locus.

To find the angle at which two surfaces of constant conical angles, ω_1 and ω_2, cut each other at any point, O, on their common edge of intersection, describe the unit sphere round O as centre. Then we have just seen that the conical angle belonging to the surface ω_1 is the area of the sphere included by a closed curve on its surface consisting of a great semicircle S_1 and an irregular curve U; and the conical angle ω_2 belonging to the other surface is the area of the sphere included between a great semicircle S_2 (having the same diameter as S_1) and the same irregular curve U. Hence $\omega_1 \sim \omega_2$ is the area of the lune included between S_1 and S_2; but S_1 and S_2 lie in the tangent planes to the surfaces ω_1 and ω_2, respectively, so that the angle, θ, between them is the angle at which the two surfaces intersect; and the area of the lune $= 2\theta$ square centimetres, if the radius of the unit sphere is 1 cm.

Hence *two surface-loci of constant conical angles* ω_1, ω_2 *for a given circuit intersect at a constant angle at all points on this circuit, the angle between them being*

$$\tfrac{1}{2}(\omega_1 \sim \omega_2).$$

316,*a*. **Line-Integrals and Surface-Integrals.** The discussion of the Conical Angles subtended at various points in space by a given circuit depends on certain theorems of integration with reference to unclosed surfaces and their bounding edges, and as the whole subject is of much importance, particularly in the theory of Electromagnetism, it is considered advisable to devote special consideration to it here.

Let ϕ (x, y, z), which we shall briefly denote by ϕ, be any function of the co-ordinates of a point in space; then if any surface (closed or unclosed) be broken up into infinitesimal elements of area and the element, dS, of area at any point be multiplied by the value of ϕ which belongs to that point, the sum of all such products, viz.

$$\int \phi dS,$$

taken all over the surface, is called the *Surface-Integral* of ϕ over the given surface.

In the same way, if any curve (closed or unclosed) be taken in space, and if it is broken up into infinitesimal elements of length,

and the element of length, ds, at any point be multiplied by the value of ϕ which belongs to that point, the sum of all such products, viz.

$$\int \phi\, ds,$$

taken all over the curve, is called the *Line-Integral* of ϕ over the given curve.

THEOREM 1. *If ϕ and ψ are any two functions of x, y (only), and if any closed plane curve be described in the plane xy, the double integral*

$$\iint \left(\frac{d\psi}{dx} - \frac{d\phi}{dy}\right) dx\, dy \qquad (\alpha)$$

taken over the area of the curve, is equal to the integral

$$\int (\phi\, dx + \psi\, dy) \qquad (\beta)$$

taken along the perimeter of the curve in the sense in which the positive axis of x should rotate in order to coincide with the positive axis of y.

Let $rpq\ldots$, Fig. 275, represent the given curve. Take the term $\iint \frac{d\psi}{dx} dx dy$ first. Now, to find this, we may first integrate with respect to x, considering y constant.

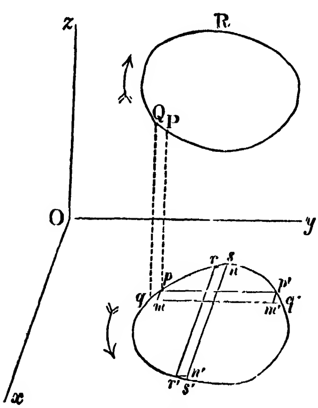

Fig. 275.

Let r be any point on the contour and rr' a line parallel to the axis of x; let s be a point on the curve infinitely near r, and ss' a line parallel to the axis of x; and let $r'n'$ and rn be perpendiculars on ss'.

Taking y, then, as constant, we are to sum $\frac{d\psi}{dx} dx dy$ over the narrow strip $srr's'$. The sum is

$$rn \,.\int \frac{d\psi}{dx} dy, \text{ or } rn(\psi' - \psi), \qquad (\gamma)$$

if ψ' and ψ are the values of ψ at r' and r, respectively.

Now observe that if we travel over the contour in the sense of the arrow, taking at each point the value of the product

$$\psi\, dy,$$

where, of course, dy is the algebraic increment of y in each

infinitesimal step, we should have in travelling from r' to s' the term

$$\psi' \times r'n',$$

and in travelling from s to r the term

$$\psi \times (-rn),$$

since the value of ψ at s may obviously be taken the same as at r. These two terms, therefore, give the sum (γ), so that the summation of ψdy over the contour will correctly give the result of the integration of the strip $rr's's$, and over all other similar strips.

In the same way, the term $-\iint \frac{d\phi}{dy}\,dxdy$ is to be found by integrating first with respect to y, considering x constant. Let, then, pp' and qq' be two indefinitely close parallels to the axis of y, enclosing a narrow strip. The summation is to be performed over this strip from p to p', so that if ϕ' and ϕ are the values of ϕ at p' and p, respectively, we have

$$dx\int\frac{d\phi}{dy}dy = pm\,(\phi' - \phi)\,;$$

$$\therefore\ -dx\int\frac{d\phi}{dy}dy = pm\,(\phi - \phi')\,; \tag{δ}$$

and in travelling over the contour in the sense of the arrow, while taking at each point the value of the product

$$\phi\, dx,$$

we should have at p the term $\phi \times pm$, and at q' the term $\phi' \times (-p'm')$, the sum of which is $pm\,(\phi - \phi')$, which is precisely (δ).

Hence, then, the area-integral (α) is equal to the contour-integral (β), which can, of course, be expressed in the form of the line-integral

$$\int\left(\phi\frac{dx}{ds} + \psi\frac{dy}{ds}\right)ds, \tag{ϵ}$$

where ds is the element of length at any point of the curve.

THEOREM 2. *If ϕ is any function of x, y, z, the co-ordinates of a point in space, and l, m, n the direction-cosines of the outward normal at any point of an unclosed surface, the integral*

$$\int\left(l\frac{d\phi}{dy} - m\frac{d\phi}{dx}\right)dS \tag{ζ}$$

taken over the surface, is equal to the integral

$$\int \phi dz \tag{η}$$

taken along the bounding edge of the surface by a motion whose projection on the plane of xy is in the sense in which the positive axis of x should rotate in order to coincide with the positive axis of y.

It must be observed that $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$ are the partial differential coefficients of ϕ with respect to x and y, and that they take no account of any variation of z—belonging, as they are supposed to do, indifferently to all points in space, and not being restricted to the (related) points which lie on the given surface.

Suppose that the co-ordinates of points on the given surface are related by the equation

$$z = f(x, y),$$

or
$$dz = p\,dx + q\,dy, \tag{1}$$

as is usual, where p and q are functions of x and y only.

Then $l = \frac{-p}{\sqrt{1+p^2+q^2}}$, $m = \frac{-q}{\sqrt{1+p^2+q^2}}$, $n = \frac{1}{\sqrt{1+p^2+q^2}}$,

and $dS = \sqrt{1+p^2+q^2}\,dx\,dy$.

Then the given integral (ζ) can be expressed in the form

$$\iint \left(q\frac{d\phi}{dx} - p\frac{d\phi}{dy}\right) dx\,dy, \tag{2}$$

which is a double integral over the area of the projection, $srpq$..., of the given surface S on the plane xy.

Now, of course, a passage from point to point of the area of this projection will correspond to a motion from one point to another on the given surface S, and will necessarily involve a variation of z in both $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$.

Denote by $\frac{\partial\phi}{\partial x}$ the total differential coefficient of ϕ with respect to x in the passage from one point on S to a neighbouring point when y remains constant but z is altered with x. Then

$$\frac{\partial\phi}{\partial x} = \frac{d\phi}{dx} + p\frac{d\phi}{dz}.$$

Similarly
$$\frac{\partial\phi}{\partial y} = \frac{d\phi}{dy} + q\frac{d\phi}{dz}.$$

Hence (2) becomes

$$\iint \left(q\frac{\partial\phi}{\partial x} - p\frac{\partial\phi}{\partial y}\right) dx\,dy. \tag{3}$$

Taking the terms of this double integral separately, we have first to integrate $q\frac{\partial\phi}{\partial x}$ with respect to x, considering y constant, i. e. to perform a summation along the strip rs'. Denote, as usual, $\frac{dq}{dx}$ by s.

Now $$\int q\frac{\partial\phi}{\partial x}dx = (q\phi)' - (q\phi) - \int s\phi dx,$$

where $(q\phi)'$ is the value of $q\phi$ at r', and $(q\phi)$ its value at r.

In a motion round the curve $srpq$... in the sense of the arrow, the term $[(q\phi)' - (q\phi)] \times r'u'$ is the same as the sum of the values of

$$q\phi \, . \, dy$$

at r' and r, as explained in Theorem 1. Hence

$$\iint q\frac{\partial\phi}{\partial x}dxdy = \int q\phi dy - \iint s\phi dxdy, \tag{4}$$

in which the single integral is one along the contour $srpq$....

Similarly $$-\iint p\frac{\partial\phi}{\partial y}dxdy = \int p\phi dx + \iint s\phi\, dx\, dy, \tag{5}$$

the single integral on the right side being taken round $srpq$... in the sense of the arrow. Hence (3) becomes simply

$$\int \phi(pdx + qdy). \tag{6}$$

But, if x, y are the co-ordinates of any point, p, on the curve $srpq$..., the point P on the edge of S, of which p is the projection, will have the same x and y, and by (1) the increment of z in passing from P to Q (of which q is the projection) is the multiplier of ϕ in (6), so that (6) is the value of

$$\int \phi dz$$

in a motion round the bounding edge PQR..., in the sense of the arrow, which was to be proved.

In the same way, of course, $\int\left(n\frac{d\phi}{dx} - l\frac{d\phi}{dz}\right)dS =$ the line-integral $\int\phi dy$ taken along the bounding edge.

We shall find it convenient to denote the operations

$$m\frac{d}{dz} - n\frac{d}{dy}, \quad n\frac{d}{dx} - l\frac{d}{dz}, \quad l\frac{d}{dy} - m\frac{d}{dx},$$

with regard to any surface the direction-cosines of whose normal are l, m, n, by the symbols

$$\partial_1, \partial_2, \partial_3.$$

THEOREM 3. *If u, v, w are any functions of x, y, z the co-ordinates of a point in space, and l, m, n the direction-cosines of the normal at any point on an unclosed surface, the integral*

$$\int\left\{l\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+m\left(\frac{du}{dz}-\frac{dw}{dx}\right)+n\left(\frac{dv}{dx}-\frac{du}{dy}\right)\right\}dS$$

taken over the surface, is equal to the integral

$$\int(u\,dx+v\,dy+w\,dz)$$

taken over the bounding edge of the surface by a motion which projects on the co-ordinate planes in the senses of positive rotation of these planes.

This follows at once from the last Theorem. For, taking the term

$$\int\left(l\frac{dw}{dy}-m\frac{dw}{dx}\right)dS,$$

we have found that it is simply $\int w\,dz$ taken along the edge. Similarly

$$\int\left(m\frac{du}{dz}-n\frac{du}{dy}\right)dS=\int u\,dx,$$

taken along this edge; &c.

This is the result that *the line-integral of any vector taken along any circuit is equal to twice the surface-integral of the normal component of its 'rotation,' or 'curl,' taken over any surface having the given circuit for a bounding edge.*

Another discussion of this Theorem will be found in Clerk Maxwell's *Electricity and Magnetism*, vol. i, Art. 24.

The result in this Theorem gives the solution of the following inverse problem:—Given the components, U, V, W, of a vector, ρ, which satisfy at all points in space the equation

$$\frac{dU}{dx}+\frac{dV}{dy}+\frac{dW}{dz}=0,$$

to determine the components of another vector, σ, such that the surface-integral of the normal component of ρ over any unclosed surface shall be equal to the line-integral of the tangential component of σ taken along the bounding edge of the given surface.

For, in order to transform the given surface-integral into a line-integral along the edge, we must have

$$lU+mV+nW\equiv\partial_1u+\partial_2v+\partial_3w,$$

that is $$\frac{dw}{dy}-\frac{dv}{dz}=U;\ \frac{du}{dz}-\frac{dw}{dx}=V;\ \frac{dv}{dx}-\frac{du}{dy}=W. \qquad (7)$$

316, *b*.] **Calculation of Conical Angles.** Let ω be the conical angle subtended by a given circuit, $PQR\ldots$, at any point A whose co-ordinates are α, β, γ. Then by (α), Art. 316, if P is any point on any surface having the circuit for edge, and dS an element of area of this surface at P,

$$d\omega = \frac{1}{r^2}\cos\psi\, dS, \tag{1}$$

where $r = AP$, $d\omega$ = conical angle subtended by dS at A, and ψ is the angle between AP and the normal to the surface at P.

Now if x, y, z are the co-ordinates of P, and l, m, n the direction-cosines of the normal,

$$\cos\psi = \frac{1}{r}\{l(x-\alpha)+m(y-\beta)+n(z-\gamma)\}. \tag{2}$$

Hence

$$\omega = \int\frac{1}{r^3}\{l(x-\alpha)+m(y-\beta)+n(z-\gamma)\}\,dS. \tag{3}$$

But since $r^2 = (x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$, we have

$$\frac{x-\alpha}{r^3} = \frac{d}{d\alpha}\left(\frac{1}{r}\right),$$

with similar values of $\frac{y-\beta}{r^3}$ and $\frac{z-\gamma}{r^3}$. Hence (3) can be written

$$\omega = \int\left\{l\frac{d}{d\alpha}\left(\frac{1}{r}\right)+m\frac{d}{d\beta}\left(\frac{1}{r}\right)+n\frac{d}{d\gamma}\left(\frac{1}{r}\right)\right\}dS. \tag{4}$$

But since α, β, γ are completely independent of all co-ordinates on the surface S, and therefore have nothing to do with the limits of integration, the symbols of differentiation with respect to them can be taken outside the integrals, and we have

$$\omega = \frac{d}{d\alpha}\int\frac{l}{r}\,dS+\frac{d}{d\beta}\int\frac{m}{r}\,dS+\frac{d}{d\gamma}\int\frac{n}{r}\,dS. \tag{5}$$

Differentiate both sides with respect to α, and observe that

$$\left(\frac{d^2}{d\alpha^2}+\frac{d^2}{d\beta^2}+\frac{d^2}{d\gamma^2}\right)\frac{1}{r} \equiv 0,$$

so that for $\frac{d^2}{d\alpha^2}$ we may write $-\left(\frac{d^2}{d\beta^2}+\frac{d^2}{d\gamma^2}\right)$. Then

$$\frac{d\omega}{d\alpha} = \frac{d}{d\beta}\left[\frac{d}{d\alpha}\int\frac{m}{r}\,dS-\frac{d}{d\beta}\int\frac{l}{r}\,dS\right]-\frac{d}{d\gamma}\left[\frac{d}{d\gamma}\int\frac{l}{r}\,dS-\frac{d}{d\alpha}\int\frac{n}{r}\,dS\right]. \tag{6}$$

Now obviously $\frac{d}{d\alpha}\left(\frac{1}{r}\right) = -\frac{d}{dx}\left(\frac{1}{r}\right)$; $\frac{d}{d\beta}\left(\frac{1}{r}\right) = -\frac{d}{dy}\left(\frac{1}{r}\right)$; &c.
Hence, first bringing the symbols of differentiation which are within the square brackets under the integral signs, (6) can be written

$$\frac{d\omega}{d\alpha} = \frac{d}{d\beta}\int\left(l\frac{d\frac{1}{r}}{dy} - m\frac{d\frac{1}{r}}{dx}\right)dS - \frac{d}{d\gamma}\int\left(n\frac{d\frac{1}{r}}{dx} - l\frac{d\frac{1}{r}}{dz}\right)dS. \quad (7)$$

Now, by Theorem 2 of the last Article, the surface-integral on which $\frac{d}{d\beta}$ operates is $\int\frac{1}{r}\,dz$ taken along the given circuit, while that on which $\frac{d}{d\gamma}$ operates is $\int\frac{1}{r}\,dy$ taken along the circuit; so that

$$\frac{d\omega}{d\alpha} = \frac{d}{d\beta}\int\frac{dz}{r} - \frac{d}{d\gamma}\int\frac{dy}{r}, \quad (8)$$

similar values holding for $\frac{d\omega}{d\beta}$ and $\frac{d\omega}{d\gamma}$.

Denoting the line-integrals $\int\frac{dx}{r}$, $\int\frac{dy}{r}$, $\int\frac{dz}{r}$ along the given circuit by F, G, H, respectively, as in Example 42, Art. 241, we have for the differential coefficients of the conical angle subtended by the given circuit at any point (α, β, γ) the equations

$$\left.\begin{aligned} \frac{d\omega}{d\alpha} &= \frac{dH}{d\beta} - \frac{dG}{d\gamma}, \\ \frac{d\omega}{d\beta} &= \frac{dF}{d\gamma} - \frac{dH}{d\alpha}, \\ \frac{d\omega}{d\gamma} &= \frac{dG}{d\alpha} - \frac{dF}{d\beta}. \end{aligned}\right\} \quad (9)$$

It is evident that the conical angle subtended by a given circuit at any point (α, β, γ) satisfies the differential equation

$$\left(\frac{d^2}{d\alpha^2} + \frac{d^2}{d\beta^2} + \frac{d^2}{d\gamma^2}\right)\omega = 0, \text{ or } \nabla^2\omega = 0. \quad (10)$$

Again, the quantities F, G, H which have reference to a given circuit and any point (α, β, γ) satisfy the equations

$$\frac{dF}{d\alpha} + \frac{dG}{d\beta} + \frac{dH}{d\gamma} = 0, \quad (11)$$

$$\nabla^2 F = 0, \quad \nabla^2 G = 0, \quad \nabla^2 H = 0. \quad (12)$$

For, the left-hand side of (11) is

$$-\int\left(\frac{d\frac{1}{r}}{dx}dx+\frac{d\frac{1}{r}}{dy}dy+\frac{d\frac{1}{r}}{dz}dz\right),$$

which, being taken along a closed curve, is zero. Hence if space were imagined to be filled with a fluid in motion, or a substance in a state of strain, its velocity components, or components of strain, at each point, A, being F, G, H, the cubical compression at every point would be zero, and the axis of resultant vortical spin at the point would be the direction in which the conical angle subtended by the circuit increases most rapidly.

Another method of calculating the conical angle subtended at a point by a circuit is the following. Let $ABCDE$ (Fig. 273) be the circuit, and O the point at which the conical angle is subtended. Then if a is the radius of the sphere described round O, the conical angle is the area of the spherical curve $abcde$ divided by a^2. Through O draw any line, Oz, meeting the surface of the sphere in z (not represented in the figure). For definiteness, suppose z to be within the part of the spherical surface which we regard as the area of $abcde$. Then the position of any point, p, within $abcde$ may be expressed by its angular distance, θ', from z, and the angle, ϕ, which the plane pzO makes with any fixed plane through Oz. These angles are the colatitude and the longitude of p. An element of spherical area at p is $a^2\sin\theta'\,d\theta'\,d\phi$, so that the strip of area of $abcde$ contained between two longitude planes including an angle $d\phi$ is

$$a^2\,d\phi\int_0^\theta \sin\theta'\,d\theta',$$

where θ is the colatitude of the point in which the arc zp intersects the contour of $abcde$.

Hence the conical angle is given by the equation

$$\omega=\int_0^{2\pi}(1-\cos\theta)\,d\phi \tag{13}$$

since ϕ runs from 0 to 2π round z.

It is, of course, quite indifferent which portion of the spherical surface (the upper or the lower) we regard as being the area of any curve traced on the sphere. If Oz is drawn so that z is in that portion of the surface which is regarded as outside the area, the longitude, ϕ, of a point within the area will not run from 0

to 2π, but from its initial value it will, after increasing and diminishing, return to this initial value, so that $\int d\phi = 0$. With an axis Oz so chosen, we should have

$$\omega = \int \cos\theta\, d\phi, \tag{14}$$

the upper and lower limits of ϕ being identical.

In the case of any *plane* circuit we obtain another expression for the conical angle subtended at any point in space. Taking the plane of the circuit as that of x, y, let (α, β, γ) be the coordinates of the point, A, at which the conical angle is required. At any point, P, in the area of the curve let the element of area be dS, and let $AP = r$. Then in (α), Art. 316, we have

$$\cos\psi = \frac{\gamma}{r}, \quad \therefore\ d\omega = \frac{\gamma}{r^3}dS. \quad \text{But } \frac{\gamma}{r^3} = -\frac{d}{d\gamma}\left(\frac{1}{r}\right),$$

therefore

$$\omega = -\frac{d}{d\gamma}\int\frac{dS}{r}. \tag{15}$$

The method of calculating ω from this equation will be understood when we come to the treatment of Potential; and it will then be seen that (15) expresses the fact that the conical angle subtended at any point by a plane curve is the same (to a factor *près*) as the component of the attraction-intensity normal to the plane of the curve exerted at the point by a uniform plane lamina bounded by the curve.

Thus for a circular curve of radius a, if $R = \sqrt{\alpha^2+\gamma^2}$ is the distance of A from the centre, and $\beta = 0$,

$$\omega = -\frac{d}{d\gamma}\int_0^{2\pi}\int_0^a \frac{r\,d\phi\,dr}{\sqrt{R^2 - 2\alpha r\cos\phi + r^2}}, \tag{16}$$

which reduces to Elliptic Integrals.

Examples.

1. Find the conical angle subtended at any point on a sphere by a given circle lying on the sphere.

Ans. Let r be the angular radius of the given circle, α = angular distance of the point considered from the pole of the circle,

$$r+\alpha = 2\sigma, \quad r-\alpha = 2\delta, \quad k^2 = \frac{\sin\alpha\sin r}{\sin^2\sigma}, \quad \text{and } n = \frac{\sin\alpha\sin r}{\cos^2\sigma};$$

then

$$\omega = 2\pi - \frac{2}{\sin\sigma}\left\{-\cos r \,.\, F(k) + \frac{\cos\delta}{\cos\sigma}\Pi(n, k)\right\},$$

where $F(k)$ is the complete elliptic integral of the first kind with modulus k, and $\Pi\,(n, k)$ the complete integral of the third kind with modulus k and parameter n.

The *complete* integral of the third kind is expressible in terms of integrals of the first and second kinds; thus

$$\frac{\cos\delta}{\sin\sigma\cos\sigma}\Pi\,(n, k)$$

$$=\frac{\pi}{2}+\left\{\frac{\cot\sigma}{\cos\delta}-E\,(k', \beta)\right\}F(k)-\{E\,(k)-F\,(k)\}\,F(k', \beta),$$

where $$k'=\frac{\sin\delta}{\sin\sigma}, \text{ and } \sin\beta=\frac{\cos\sigma}{\cos\delta}.$$

2. Show that the conical angle subtended at any point, A, by a circuit is the line-integral along the circuit of the tangential component of a vector whose magnitude at each point, P, of the circuit is

$$\frac{1}{r}\,\frac{\cos\theta\cos\lambda}{\sin^2\theta},$$

the vector being perpendicular to AP in the plane of AP and the tangent at P, $r=AP$, θ is the angle made by AP with a fixed line, and λ the angle made with this line by the normal to the plane of AP and the tangent at P.

3. Find the conical angle subtended at any point, P, in space by two intersecting right lines OA, OB, their extremities A and B being both at infinity.

Ans. If ϕ and ϕ' are the angles between the plane AOB and the planes containing P and the lines OA, OB, and $\alpha=\angle AOB$

$$\omega=\pi-\phi-\phi'+\cos^{-1}(\sin\phi\sin\phi'\cos\alpha-\cos\phi\cos\phi'). \qquad (1)$$

When $\alpha=0$, the plane from which ϕ and ϕ' are reckoned is indeterminate, but in this case $\phi+\phi'$ is π, so that ω is constant whatever be the position of P. When $\alpha=\pi$, $\phi=\phi'$, and $\omega=2\pi-2\phi$, which is indeterminate and may be taken as 2ϕ simply, where ϕ is the longitude of P with reference to any fixed plane through the infinite line AOB.

If $t=\tan\phi$, $t'=\tan\phi'$, the equation of the surface locus of constant conical angle is

$$(t+t')\sin\omega+tt'(\cos\omega-\cos\alpha)+2\sin^2\tfrac{1}{2}\,\omega=0. \qquad (2)$$

To all points on the same right line OP through O belongs the same value of ω; moreover, this equation shows that the planes determining any given angle ω can be represented in pairs by the points of an involution system.

The surfaces of constant conical angles are cones of the second degree whose equations are easily found from (2). For, if OA is taken as axis of x, and the plane AOB as that of xy, we find for the locus

$$2\,y\,(x\sin\alpha-y\cos\alpha)\sin^2\tfrac{1}{2}\,\omega+z\,(x\sin\alpha+2\,y\sin^2\tfrac{1}{2}\,\alpha)\sin\omega$$
$$+z^2(\cos\omega-\cos\alpha)=0; \qquad (3)$$

or, taking the internal and external bisectors of the angle AOB as axes of x and y,

$$\sin^2\frac{\omega}{2}(x^2\sin^2\frac{\alpha}{2}-y^2\cos^2\frac{\alpha}{2})+z^2(\sin^2\frac{\alpha}{2}-\sin^2\frac{\omega}{2})+xz\sin\frac{\alpha}{2}\sin\omega=0. \quad (4)$$

The conical angle is a measure of the Magnetic Potential at any point due to a current in the given circuit; hence the case $\alpha = 0$ corresponds to a current returning on itself, which, of course, produces no effect at any point; while $\alpha = \pi$ corresponds to a straight current of (practically) infinite length.

4. In the case of any plane triangular circuit whose angles are A, B, C, prove the following construction for points on the surface-locus of constant conical angle, ω:—

From any point, O, on a sphere draw arcs, OL, OM, ON, of three great circles including between them angles equal to $\pi-C$, $\pi-A$, $\pi-B$; then describe any spherical triangle, LMN, whose vertices lie on these arcs, such that the sum of its sides $= 2\pi-\omega$; the angles subtended at the centre of the sphere by the arcs OL, OM, ON will be the inclinations to the plane of the triangle ABC of planes drawn through its sides BC, CA, AB intersecting in a point, P, at which the conical angle is ω.

Thus, then, to find the point on any given line, AP, drawn through A at which the triangle subtends ω, we take two given points M, N on two of the arcs and find the point, L, on the third such that $LM+LN$ is a given quantity. There are two solutions, since, given base and sum of sides of a spherical triangle, the locus of the vertex is a sphero-conic.

5. Show that the *complete* solution of equations (7), Art. 316 (a), from u, v, w will necessarily be indeterminate.

(To any values found for u, v, w may be added terms $\frac{dP}{dx}, \frac{dP}{dy}, \frac{dP}{dz}$, where P is any function of x, y, z which has a single (unambiguous) value for given values of x, y, z.)

6. For the conical angle subtended by a given plane circle at any point in space, show that the angles $\alpha, r, \sigma, \delta$ in Example 1 can be exhibited by a plane construction.

317.] **Attraction of a Thin Uniform Straight Bar.** Let the line AB (Fig. 276), represent a straight bar the area of whose transverse section is k square centimetres, this area being very small; let the mass of the bar be ρ grammes per cubic centimetre of substance; let P be the position of a mass of 1 gramme supposed to be condensed into an infinitely small volume. It is required to find the magnitude and direction of the attraction of the bar on the particle at P.

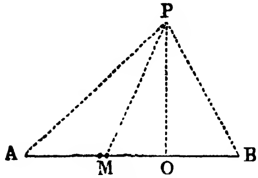

Fig. 276.

Draw PO perpendicular to AB; take any point, M, on AB; let $OM = s$, and consider the attraction on P of the elementary length ds at M. The mass at P being 1 gramme, and the mass of ds being $\rho \,.\, kds$, if γ is the constant of gravitation (Art. 315), the attraction of ds on P is, in dynes,

$$\gamma \frac{k\rho ds}{PM^2}. \tag{1}$$

This force acts in the line PM. Resolve it into a component along PO and a component perpendicular to PO. Let

$$\phi = \angle OPM;\ PM = r;\ PO = p;$$

and let these components be dY and dX, respectively.

Then
$$dX = \frac{\gamma k\rho}{r^2} \sin\phi \,.\, ds,$$

$$dY = \frac{\gamma k\rho}{r^2} \cos\phi \,.\, ds.$$

But $s = p\tan\phi$, $\therefore\ ds = p\sec^2\phi d\phi$, and $r = p\sec\phi$. Hence

$$dX = \frac{\gamma k\rho}{p} \sin\phi d\phi;$$

$$dY = \frac{\gamma k\rho}{p} \cos\phi d\phi.$$

Then, obviously, if $\angle OPA = \alpha$, and $\angle OPB = \beta$, and if X and Y are the total component attractions parallel and perpendicular to BA produced by all the elements of the bar, we have

$$X = \frac{\gamma k\rho}{p}\int_{-\beta}^{\alpha} \sin\phi d\phi = \frac{\gamma k\rho}{p}(\cos\beta - \cos\alpha), \tag{2}$$

$$Y = \frac{\gamma k\rho}{p}\int_{-\beta}^{\alpha} \cos\phi d\phi = \frac{\gamma k\rho}{p}(\sin\beta + \sin\alpha). \tag{3}$$

If the resultant attraction on P makes the angle ψ with PO, we have $\tan\psi = \dfrac{X}{Y} = \tan\dfrac{\alpha-\beta}{2}$, $\therefore\ \psi = \dfrac{\alpha-\beta}{2}$, which shows that *the resultant, R, bisects the vertical angle APB.* Also

$$R = \frac{2\gamma k\rho}{p}\sin\frac{APB}{2} \text{(dynes)}. \tag{4}$$

We may also notice the simple fact that *the attraction of the bar AB on P is the same in all respects as the attraction of a circular arc having P as centre with radius PO, this arc being terminated by the lines PA and PB*, the density and area of

transverse section of this arc being the same as those of the given bar. For, if N is the point on AB distant ds from M, and if the lines PM and PN meet the circular arc in m and n, the attractions of MN and mn on P are exactly the same, because if from M a perpendicular MQ is drawn to PN, we have

$$MN = \frac{MQ}{\sin PMO} = \frac{MQ \,.\, PM}{PO} = \frac{mn \,.\, PM^2}{Pm \,.\, PO} = \frac{mn \,.\, PM^2}{Pm^2};$$

therefore $\dfrac{MN}{PM^2} = \dfrac{mn}{Pm^2}$, and the attractions of corresponding elements of the bar AB and the circular arc are the same.

The attraction of the particle P on the bar is R exactly reversed.

For an *infinitely long bar*, or one so long that the distances of P from its extremities are each very great compared with the distance, PO, of P from the bar, the attraction is

$$\frac{2\gamma k\rho}{p}, \tag{5}$$

since the angle APB is in this case π.

If the law of attraction is not that of the inverse square of distance, let the attraction of the element $k\rho ds$ at M on the unit mass at P be expressed by the law

$$k\rho ds \times \lambda f'(r),$$

where λ is a constant and $f'(r)$ any function of the distance PM.

Then, if $PA = r_2$ and $PB = r_1$, we easily find

$$X = \lambda k\rho \left[f(r_2) - f(r_1) \right], \tag{6}$$

$$Y = \lambda k\rho \int_{r_1}^{r_2} \frac{f'(r)\,dr}{\sqrt{r^2 - p^2}}. \tag{7}$$

The expression (5) brings us back to the observation made in Art. 315 with regard to the application of the law of inverse square to particles separated by an infinitely small distance; for it would appear from this expression that if $p = 0$, or the attracted particle is on the surface of the bar, the attraction is ∞: a result which is obviously absurd. The whole of our investigation depends on the assumption that every point in the element ds of length at M is at the same distance, r, from P. Now if P is in contact with the surface, the particles of the bar in the normal section at P are at all distances ranging from zero to the

diameter of the bar from P, so that we cannot expect our result to hold for this case. In fact, k, the area of the normal section, ought in this case to be *infinitely* small, and then the expression (5) is indeterminate. To find what is really the intensity of attraction at a point on the surface of the bar, we must consider this latter as a cylinder of finite radius, a, and break it up into slender filaments in such a way that a filament to which P is infinitely close is one of infinitely small section. Such a mode of breaking up the bar is obtained by a polar resolution. Thus: draw the normal section through P; take any point Q in the area of this section, let $PQ = r$, and take the usual polar element, $rdrd\theta$, of area at Q. Consider now the attraction at P due to the filament of the bar, parallel to its axis, which stands on this element of area. It is clear that the filaments are now taken in such a way that when the distance of P from one of them vanishes, so does the transverse section of the filament.

For greater generality, let P be assumed outside the bar at a distance c from its centre, O; let the transverse section be circular and the length of the bar *practically* infinite, i.e. P is so close to the surface, that for each filament the angle APB may be taken as π.

The attraction of the filament at Q on a unit mass condensed at P is $\frac{2\gamma\rho \,.\, rd\theta dr}{r}$, or $2\gamma\rho d\theta dr$; and since PO is the direction of the resultant, we resolve this along PO; thus we have $2\gamma\rho\cos\theta\, d\theta dr$, where $\theta = \angle QPO$. Integrating this first with respect to r between the points at which the line PQ enters and leaves the circular section, we have

$$4\gamma\rho\sqrt{a^2 - c^2\sin^2\theta}\cos\theta d\theta,$$

as the contribution of the wedge of bars corresponding to the angle θ. The extreme values of θ are $\pm\sin^{-1}\frac{a}{c}$, so that a further integration gives

$$\frac{2\pi\gamma\rho a^2}{c}$$

for the attraction of a cylindrical bar at a point near its surface, the length of the bar being very great compared with its diameter. Now if the position of the attracted point is on the surface, $c = a$, and the attraction is

$$2\pi\gamma\rho a.$$

318.] Uniform Thin Circular Plate. Consider a circular plate of uniform density (ρ grammes per cubic centimetre of substance) and very small uniform thickness (τ centimetres); and let 1 gramme mass be condensed into a point P situated on the axis of the plate, i. e. a line drawn through O, the centre of the plate, perpendicular to the plane of the plate. It is required to find the attraction of the plate on the particle at P. Let a (centimetres) be the radius of the plate, and let $OP = z$ (centimetres). Take any point, Q, in the plane of the plate and describe a circle with centre O and radius OQ $(= r)$. Concentric with this describe another circle of radius $r + dr$, so that a narrow strip of area is included between these circles. We may in this way break up the plate into an infinitely great number of circular strips; the mass of the typical strip is $2\pi\rho\tau r dr$ grammes, and all points in the strip are at the same distance, PQ, or $\sqrt{z^2+r^2}$, from P. Also, if ϕ is the angle OPQ, since the resultant force exerted on P by the strip is obviously along PO, this resultant is

$$\gamma \cdot \frac{2\pi\rho\tau r dr}{z^2+r^2}\cos\phi, \text{ or } 2\pi\gamma\rho\tau \sin\phi d\phi,$$

since $r = z\tan\phi$, γ being the constant of gravitation.

If α is the semi-angle of the cone whose vertex is P and base the rim of the plate, ϕ ranges from 0 to α, so that

$$R = 2\pi\gamma\rho\tau\,(1-\cos\alpha), \tag{1}$$

in dynes. This can be written $2\pi\gamma\rho\tau\left(1-\dfrac{z}{\sqrt{z^2+a^2}}\right)$.

From this expression is deduced a result of great importance in Electrostatics. Suppose the attracted particle P to be very close to the plate, *at the same time that the latter is infinitely thin compared with the distance of P*—this supposition being obviously necessary if we are to assume that all the particles in the body of the plate and included in a circular strip are equidistant from P. Then lines drawn from P to the rim of the plate are practically at right angles to OP, so that $\alpha = \frac{1}{2}\pi$, and

$$R = 2\pi\gamma\rho\tau \text{ (dynes)}, \tag{2}$$

and the result is independent of the radius of the plate. Thus, if P is at a distance of 1 millimetre from the centre of such a plate, the attraction on P is practically the same whether the radius of the plate is infinitely great or only 1 decimetre.

Again, the expression (1) shows that any two uniform plates of the same substance and of the same small thickness will exert equal forces on P if they are cut from the cone having P for vertex (their planes being parallel). Hence any two frustums of equal thickness, h, however great, cut from this cone will equally attract the particle P at its vertex, the magnitude of the force being

$$2\pi\gamma\rho h(1-\cos\alpha).$$

The result holds also in the case of an oblique cone standing on any plane base whatever, the attracted particle P being at its vertex. To prove this geometrically we have merely to show that if two plates of the same thickness, each parallel to the base, be taken anywhere in the cone, they exert equal attractions on a particle at the vertex. Through the vertex P draw an infinite number of rays forming a very slender cone which intercepts on the two plates two small similar areas, dS and dS', at the points M and M', suppose. Then the attraction of dS on P is $\frac{\gamma\rho\tau dS}{PM^2}$, and that of dS' is $\frac{\gamma\rho\tau dS'}{PM'^2}$, these forces being coincident in the line PMM'. But since the contours of dS and dS' are similar curves, $\frac{dS}{dS'}=\frac{PM^2}{PM'^2}$; therefore these attractions are equal. Similarly for all other pairs of corresponding elements of the plates.

This put into the usual form of analysis would be as follows: Let (r, θ, ϕ) be the radius vector from P to M, the colatitude and longitude (Art. 175) of M. Then the element of volume at M may be taken as $r^2\sin\theta dr d\theta d\phi$, and the attraction on P of the element of mass included is $\gamma\rho\sin\theta\, dr\, d\theta\, d\phi$, and this depends only on dr and not on r (so long as θ and ϕ are constant); hence the attractions of the elements at M and M' above are equal, since these points have obviously the same θ and ϕ.

319.] **Uniform Spherical Shell.** Suppose a thin shell of attracting matter of uniform density, ρ grammes per cubic centimetre, to be contained between two very close concentric spheres. Then *this shell exercises no resultant attraction on any internal particle.* For, let P' be the position of any internal particle, and through P' draw a pencil of rays, $QP'Q'$, $RP'R'$, &c., forming a very slender cone; then if a ray RR' meets in the inner sphere in r and r', the lengths Rr and $R'r'$ are equal; hence the spherical surfaces at Q cut off a frustum

whose thickness is equal to that of the frustum cut off at Q', and by Art. 318 the attractions of these frustums on the particle P' at their common vertex are equal and opposite. Hence the attractions of these elements of the shell destroy each other at P', and similarly all the vertically opposite elements of the shell cut off in the same way annul each others' effects at P'.

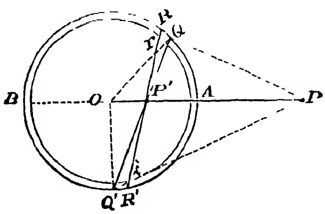

Fig. 277.

The resultant force on the particle is therefore zero.

Precisely the same result holds for an ellipsoidal shell bounded by two very close concentric and *similar* ellipsoids, since the intercepts Rr, $R'r'$ made by the shell on any line cutting its bounding surfaces are equal. This proof is given by Newton, Cor. 3, Prop. 91, Book I. These results we shall find useful in Electrostatics—in which occurs the general problem: Given the outer bounding surface of a shell, find the law of its thickness (or, in other words, find its inner bounding surface) so that its resultant attraction on every internal particle shall be zero. The simple result in the case of surfaces of the second degree, that the inner surface is one concentric with and similar to the outer, is due to the fact that they have diametral planes which bisect all parallel chords.

If the law of attraction is not that of the inverse square, let it be expressed by $\lambda \dfrac{f(r)}{r^2}$, and consider the narrow belt of the shell which is generated by the revolution of the element of arc QR about OP'. Let $OP' = c$, $P'Q = r$, $a =$ radius of shell; then the area of this strip $= 2\pi \dfrac{a}{c} r dr$; for in the usual notation it $= 2\pi y ds$, or $2\pi a^2 \sin\theta d\theta$, where $\theta = \angle QOA$, and

$$r^2 = a^2 - 2ac\cos\theta + c^2, \quad \therefore\ r dr = ac\sin\theta d\theta,$$

so that if $dS =$ area of belt,

$$dS = 2\pi \frac{a}{c} r dr, \tag{A}$$

and the mass of this belt $= 2\pi\rho\tau \dfrac{a}{c} r dr$, where $\tau =$ thickness of shell.

Every particle of this strip is at the distance r from P', and its resultant attraction on P' (which is in the direction OP) is

$$2\pi\lambda\rho\tau\frac{a}{c}r\,dr.\frac{f(r)}{r^2}.\cos QP'P, \text{ which is } \frac{\pi\lambda a\rho\tau}{c^2}.\frac{a^2-c^2-r^2}{r^2}f(r)dr.$$

Hence, if R is the resultant attraction at P',

$$R = \frac{\pi\lambda a\rho\tau}{c^2}\int_{a-c}^{a+c}\frac{a^2-c^2-r^2}{r^2}f(r)dr. \quad (1)$$

When the law of attraction is that of the inverse square, $f(r)$ is constant, and the value of the integral is zero.

From this expression can be deduced a result which is fundamental in Electrostatics—viz. *the law of the inverse square is the only law of attraction for which a spherical shell of uniform thickness and density will produce no resultant attraction on any internal particle.*

For, whatever $a+c$ and $a-c$ may be, i. e. wherever P' is situated inside, the definite integral must vanish identically. Denote $a+c$ by m and $a-c$ by n. Then for all values of m and n,

$$\int_n^m \frac{mn-r^2}{r^2}f(r)dr = 0.$$

Differentiating this with regard to m and n, successively,

$$\frac{n-m}{m}f(m)+n\int_n^m\frac{f(r)}{r^2}dr = 0,$$

$$\frac{n-m}{n}f(n)+m\int_n^m\frac{f(r)}{r^2}dr = 0.$$

Hence $f(m)=f(n)$, whatever m and n may be; i. e. $f(r)$ must be absolutely constant, so that the law of attraction is that of the inverse square.

For a particle placed at any external point, P, the attraction (for the law of the inverse square) is the same as if the shell were condensed into a particle at its centre.

This may be shown in several ways. Thus, take the inverse of P with respect to the spherical surface; let this point be P', that is, $OP\times OP' = OQ^2 = a^2$. From this equation it follows that the triangles POQ and QOP' are similar, and therefore

$$\frac{QP}{QP'}=\frac{D}{a}, \quad (\alpha)$$

where $D = OP$ and $a = OQ$; that is, *the ratio of the distances of all points on the sphere from P and P' is constant.* Again,

from the similarity of these triangles $\angle QPO = \angle P'QO$; similarly, $\angle Q'PO = \angle P'Q'O$; *therefore the angle QPQ' is bisected by PO.*

Denote QP by r and QP' by r'. Let $d\omega =$ the conical angle subtended at P' by the elements of surface of the sphere cut off at Q and at Q' by a thin cone of rays $QP'Q'$, $RP'R'$, ... Then (Art. 316) the area of the element of surface at Q is $r'^2 \sec OQP' . d\omega$, and the attraction of the mass of this element on a unit (gramme) mass at P is $\gamma\rho\tau\frac{r'^2}{r^2}\sec OQP' d\omega$ acting in PQ (γ being the constant of gravitation). This, by (α), is $\gamma\rho\tau\frac{a^2}{D^2}\sec OQP'.d\omega$. The attraction on P produced by the element at Q' is similarly $\gamma\rho\tau\frac{a^2}{D^2}\sec OQ'P'.d\omega$, and these two expressions are identical, i.e. P is equally attracted by the corresponding elements at Q and Q'. The resultant of these forces acts in PO and is equal to

$$2\gamma\rho\tau\frac{a^2}{D^2}d\omega.$$

The sum of all such forces is obtained by summing $d\omega$ from 0 to 2π. Hence the resultant attraction

$$R = \gamma . \frac{4\pi\rho\tau a^2}{D^2} \qquad (\beta)$$

$$= \gamma . \frac{\text{mass of shell}}{D^2},$$

which is exactly the same as if the shell were condensed into an infinitely small particle at its centre.

To deduce the result analytically, break up the shell, as before, into strips formed by the revolution of elements of length, QR, ... about OP. The area of such an element $= 2\pi\frac{a}{D}r\,dr$, where $r = QP$; and the attraction of the element of mass contained within this ring on the unit (gramme) mass at P is

$2\pi\gamma\rho\tau\frac{a\,dr}{D\,r}\cdot\cos QPO$, i.e. $\frac{\pi\gamma\rho\tau a}{D^2}\cdot\frac{r^2+D^2-a^2}{r^2}dr$; therefore

$$R = \frac{\pi\gamma\rho\tau a}{D^2}\int_{D-a}^{D+a}\frac{r^2+D^2-a^2}{r^2}dr$$

$$= \gamma . \frac{4\pi\rho\tau a^2}{D^2}.$$

If the law is not that of the inverse square, but expressed by $\lambda \frac{f(r)}{r^2}$, we have

$$R = \frac{\pi \lambda \rho \tau a}{D^2} \int_{D-a}^{D+a} \frac{r^2 + D^2 - a^2}{r^2} f(r)\, dr, \tag{2}$$

the limiting values of r in these integrals being PA and PB, i. e. $D-a$ and $D+a$.

To determine the laws of attraction for which the attraction of a uniform spherical shell on any external particle is the same as if the shell were condensed into an infinitely small particle at its centre. We know from Art. 23 (vol. i) that this is the case for any material body if the law of attraction be that of the direct distance; and we have just proved that for a uniform spherical shell the result holds for the Newtonian law. We shall now prove that these two are the only laws.

Denoting $D+a$ by m, and $D-a$ by n, and observing that if the shell may be condensed into a particle of mass $4\pi\rho\tau a^2$ at its centre, the value of R must be $4\pi\lambda\rho\tau a^2 \cdot \frac{f(D)}{D^2}$, we have from (2)

$$\int_n^m \frac{r^2 + mn}{r^2} f(r)\, dr = 2(m-n) f\left(\frac{m+n}{2}\right). \tag{3}$$

Dividing out by $m-n$ and differentiating with respect to m and n successively, we have

$$\frac{m+n}{m(m-n)} f(m) - \frac{1}{(m-n)^2} \int_n^m \frac{n^2 + r^2}{r^2} f(r)\, dr = f'\left(\frac{m+n}{2}\right),$$

$$-\frac{m+n}{n(m-n)} f(n) + \frac{1}{(m-n)^2} \int_n^m \frac{m^2 + r^2}{r^2} f(r)\, dr = f'\left(\frac{m+n}{2}\right);$$

therefore by subtraction,

$$(m^2 - n^2)\left[\frac{f(m)}{m} + \frac{f(n)}{n}\right] = \int_n^m \frac{m^2 + n^2 + 2r^2}{r^2} f(r)\, dr.$$

Differentiating again successively, and eliminating $\int_n^m \frac{f(r)}{r^2} dr$ from the two equations, we have simply

$$\frac{f'(m)}{m^2} = \frac{f'(n)}{n^2},$$

which must hold whatever m and n may be. Hence

$$f'(r) = Cr^2,$$

where C is a constant. If $C = 0$, $f(r)$ is constant, and we have the law of inverse square, as before. If C is not zero,

$$f(r) = \tfrac{1}{3}Cr^3, \quad \therefore \frac{f(r)}{r^2} \propto r,$$

and we have the law of the direct distance. These, therefore, are the only laws for which the result holds.

320.] **Solid Homogeneous Sphere.** Instead of a spherical shell, let Fig. 277 represent a solid sphere, and consider its attraction on a unit mass condensed at P. Imagine the sphere broken up into an infinite number of infinitely thin concentric spherical shells. Then each of these attracts P as if it were condensed into a particle at O. Hence the whole sphere may be considered as condensed into a particle of mass $\frac{4}{3}\pi\rho a^3$ at O, and if $R =$ the resultant force on the unit mass at P,

$$R = \gamma \cdot \frac{4\pi\rho a^3}{3D^2}. \tag{α}$$

If the attracted particle is inside the sphere, at P', all those shells which lie outside the sphere described with centre O and radius OP' may be rejected, since none of them produce any *resultant* * effect on P'; so that the whole force

$$= \gamma \frac{\text{mass of sphere with radius } OP'}{OP'^2},$$

or

$$R = \gamma \cdot \tfrac{4}{3}\pi\rho D', \tag{β}$$

where $D' = OP'$, i. e. *inside a homogeneous solid sphere the attraction varies as the distance of the attracted particle from the centre.*

Also any two solid homogeneous spheres attract each other as if each were condensed into a single particle at its centre. If, then, m and m' are their masses, and if c is the distance between their centres, the magnitude of their mutual attraction is

$$\gamma \cdot \frac{mm'}{c^2}. \tag{γ}$$

321.] **Value of the Constant of Gravitation.** We are now

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* To be carefully distinguished from the *pressure* effect which is produced at all internal points, and which is very great at great depths. The whole surface of a particle may be subject to great intensity of pressure, while the *resultant* force on the particle may be zero.

in a position to find γ, the absolute constant of gravitation. Let the two attracting spheres be the earth (assumed homogeneous and spherical) and a small particle whose mass is 1 gramme. The following data * relating to the magnitude and density of the earth may be assumed as approximately correct: the radius of the earth is 637×10^6 centimetres; the mass of the earth is 614×10^{25} grammes (its mean density, ρ, being 5·67 grammes per cubic centimetre); the weight of 1 gramme mass at the surface of the earth is 981 dynes. Hence, putting $R = 981$, $\rho = 5{\cdot}67$, $D' = 637 \times 10^6$ in (β), or using the expression (γ) and making $m = 614 \times 10^{25}$, $m' = 1$, $c = 637 \times 10^6$, we find

$$\gamma = \frac{1 \text{ dyne}}{1543 \times 10^4}.$$

Now a dyne being, roughly, the weight of a milligramme, we see how extremely small a magnitude is the constant of gravitation, at least, in our region of Space; for it is conceivable that in enormously distant portions of the Universe the values of γ may be different.

322.] **Sudden Change of Attraction through a Shell.** Let P and Q (Fig. 278) be two points on the normal to any thin shell at opposite sides of the surface. Suppose a unit (gramme) mass condensed at P, and regard the attraction of the shell on this particle as produced by a small circular plate just below P, and the remainder of the surface. Consider, similarly, the attraction of the shell on a unit mass at Q. Now the attractions at P and Q produced by the portion of the shell obtained by omitting the small circular plate above-mentioned are sensibly the same in magnitude and line of action. Each of these attractions may be represented by $\bar{f}$, regarded as a vector.

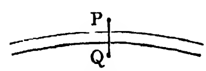

Fig. 278.

But it has been shown (Art. 318) that the attraction of the small plate on the unit mass at P is

$$2\pi\gamma\rho\tau \text{ dynes},$$

acting in the normal from P to Q; while the attraction of this plate on Q is this force exactly reversed in direction.

* See Everett's *Units and Physical Constants*, chap vi.

Denote this force on Q by the vector $\bar{n}$. Then the forces on P and Q are, vectorially,

$$\bar{f}-\bar{n} \text{ and } \bar{f}+\bar{n},$$

respectively.

If the shell is such that it exercises no resultant attraction at Q, $\bar{f}+\bar{n}=0$, and the resultant attraction on P is normal to the surface and equal to $-2\bar{n}$, i. e. to

$$-4\pi\gamma\rho\tau,$$

considered as acting in the sense of the *outward-drawn* normal, QP. Numerically, of course, the force on P is $+4\pi\gamma\rho\tau$, acting in the sense, PQ, of the inward-drawn normal.

323.] **Force-intensity.** To save circumlocution, we shall define the *force-intensity* exerted by any attracting mass at any point P as *the force exerted by the given mass on a gramme mass condensed into a point at* P.

If instead of having 1 gramme mass at P, we have a particle whose mass is dm grammes, and if the given mass attracts it with a force of dF dynes, the force-intensity at P is

$$\frac{dF}{dm}.$$

Thus the force-intensity at P of the small circular plate in last Article is $2\pi\gamma\rho\tau$ (inwards), which will be in dynes if ρ is the density of the shell at P in grammes per cubic centimetre, τ is the thickness of the shell in cms., and γ is the constant of gravitation as defined in Art. 321.

324.] **Surface-integral of Normal Force-intensity.** If round any particle, dm, of matter attracting according to the law of the inverse square any closed surface whatever be described, the surface-integral of the normal force-intensity produced by the particle (the integration being taken over this surface) is equal to $4\pi\gamma \cdot dm$; and if the surface is described so that the particle is outside it, the surface-integral is zero.

Begin with the latter case. Let O (Fig. 273) be the position of the attracting particle of mass dm grammes, and let the surface represented be any one whatever. Then the force-intensity at P_1 is $\gamma \dfrac{dm}{OP_1^2}$; the component of this along the outward normal is $\gamma \dfrac{dm}{OP_1^2} \cos OP_1 n_1$; and if dS_1 is the element of area

of the surface at P_1, we have $\gamma \frac{dm}{OP_1^2} \cos OP_1 n_1 dS_1$ for the element of the surface-integral in question. But this is simply $\gamma dm \, . \, d\omega$, where $d\omega$ is the conical angle subtended at O by dS_1. Hence, if at any point on the given surface N is the magnitude of the normal component of the force-intensity and dS is the element of area, we have

$$\begin{aligned} \int NdS &= \gamma dm \int d\omega \\ &= 0, \end{aligned} \tag{1}$$

since, as explained in Art. 316, the total conical angle subtended at any external point by the elements of any closed surface is zero.

If O is internal to the surface, the whole of the investigation remains unaltered, but $\int d\omega$ is now 4π, so that for any internal particle, dm,

$$\int NdS = -4\pi\gamma dm. \tag{2}$$

If instead of a single particle we have any number of them, all external to the given closed surface, and forming a mass which we may denote by M_e, we shall have (1) still holding, N being the normal component of the force-intensity due to the attraction of the whole mass M_e; and if inside the surface there is any mass M_i distributed in any way whatever, we have

$$\int NdS = -4\pi\gamma M_i, \tag{3}$$

γ being the constant of gravitation, having in the C. G. S. system the value given in Art. 321.

If attracting matter can be spread as an infinitely thin layer *on* the surface, and the total mass thus spread be M_0, we should have

$$\int NdS = -2\pi\gamma M_0, \tag{4}$$

N being the normal force-intensity at any point due to the action of M_0. This is obvious by Art. 316, since for any point on the surface $\int d\omega = 2\pi$.

The expression $\int NdS$ is sometimes described as the *normal flux of force through the surface outwards*. It is to be carefully noted that N has been measured at all points on the surface along the outward-drawn normal. If at any point it really acts inwards, it is to be considered as *negative* at this point.

Many results in Electrostatics depend on the theorems expressed by (1), (3), (4). These theorems are due to Gauss, and

are given in his famous paper on forces varying inversely as the square of the distance (Taylor's *Scientific Memoirs*, vol. iii, Part X).

325.] **General Components of Attraction.** Let there be any attracting body the matter of which attracts according to any law of the distance—suppose $\phi(r)$—and consider its attraction on a unit particle condensed into an infinitely small volume at any point, P, which may be either inside the attracting mass or in void space.

Let the co-ordinates of P referred to any fixed rectangular axes be (x, y, z); let P' be any point in the attracting mass, its co-ordinates being (x', y', z'); let ρ be the density of the matter at P', so that in a small parallelepiped cut out in the usual manner at P' the mass is $\rho dx'dy'dz'$; let r be the distance PP'. (We may, for definiteness, suppose these quantities to be measured in the units of the C.G.S. system.) Then the attraction of the element at P' on the condensed gramme at P is $\rho\phi(r)dx'dy'dz'$, acting in the sense $\overline{PP'}$, and the component of this parallel to the axis of x, in the positive sense of this axis, is

$$-\rho\phi(r).\frac{x-x'}{r}dx'dy'dz'.$$

Hence, if X, Y, Z denote the total components of the 'attraction-intensity' (see Art. 323) at P parallel to the axes, in their positive senses,

$$X = -\iiint \rho\phi(r)\frac{x-x'}{r}dx'dy'dz', \tag{1}$$

or simply,

$$= -\int \phi(r)\frac{x-x'}{r}dm, \tag{2}$$

with exactly similar values of Y and Z, the integrations being extended to all points, P', of the attracting mass, of which in the more succinct form (2) dm is the element of mass.

When P is within the attracting mass, the term $\frac{x-x'}{r}$ assumes the form $\frac{0}{0}$ for all the points P' in the immediate vicinity of P, and though the distances of P from some *points* in the mass are zero, we must not conclude that the attraction is infinite, because, as we have pointed out at the very beginning (Art. 315), a law of attraction according to a function of the *distance* between

two particles cannot be logically enunciated, or even conceived, except on the supposition that the dimensions of such particles are infinitely small compared with the (mean) distance between them.

As a matter of fact—and it is one of considerable importance to understand—the law of the inverse square leads to no such result as an infinite attraction when the position of the attracted particle is within the attracting mass; but the Cartesian expressions (1), (2) do not immediately show this. We shall show it by taking the elements, dm, of mass in polar co-ordinates.

Taking the position of the attracted particle P as pole, let (r, θ, ϕ) be the usual polar co-ordinates at P'. Then the element of mass at P' is $\rho r^2 \sin\theta dr\, d\theta d\phi$ (Art. 175), so that the attraction along $\overline{PP'}$ is $\rho r^2 \phi(r) \sin\theta\, dr\, d\theta\, d\phi$; hence

$$X = \iiint \rho r^2 \phi(r) \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi, \quad (3)$$

$$Y = \iiint \rho r^2 \phi(r) \sin^2\theta \sin\phi\, dr\, d\theta\, d\phi, \quad (4)$$

$$Z = \iiint \rho r^2 \phi(r) \sin\theta \cos\theta\, dr\, d\theta\, d\phi, \quad (5)$$

Now, for the law of Newton, $\phi(r) = \frac{\gamma}{r^2}$, so that

$$X = \gamma \iiint \rho \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi, \quad (6)$$

with similar values of Y and Z; and even when $r = 0$, X contains no infinite term.

If, however, the attraction between two particles increased according to a law more rapid than the inverse square, the attraction-intensity at any internal point would be infinite.

For, if $\phi(r) = \frac{\mu}{r^3}$, we shall have the term $\frac{\rho}{r}\sin^2\theta\cos\phi\, dr\, d\theta\, d\phi$ in the value of X, and this becomes ∞ for the particles P' immediately in contact with P. This supposes the mass of P fixed and finite—1 gramme, suppose. But if the particle at P is itself of infinitely small mass, the infinite value of the attraction (no longer attraction-intensity) disappears.

As explained in the chapter on Centres of Mass, it is not necessary to take in all cases infinitesimal elements of the third order in breaking up the attracting mass. According to the shape and law of density of the attracting body, we may take as elements, circular plates, thin bars, rings, &c., as will be illustrated in the following examples.

Examples.

1. Whatever may be the law of attraction, the force-intensity exerted by the smaller of two concentric solid homogeneous spheres at any point on the surface of the larger is to the force-intensity exerted by the larger at any point on the surface of the smaller in the ratio (radius of smaller)^2 : (radius of larger)^2.

Draw any radius OP meeting the surface of the larger in P and that of the smaller in p, O being the common centre. Draw a chord, ab, of the smaller parallel to OP; at a and b take equal and similar very small elements of area, each ds; draw lines from the various points of ds at a to the corresponding points of ds at b; we thus have a uniform bar of the substance of the smaller sphere lying along ab. Draw lines from O to all the points on the contour of ds at a; we thus get a slender cone; produce this cone outwards to intersect the surface of the outer sphere—at A, suppose—and let dS be the element of surface of the outer intercepted by this cone; draw similarly a cone with vertex O having ds at b for base, and let this intercept at B on the outer an element of area dS. Joining the points on the contour of dS at A to the corresponding points of dS at B, we have a bar, AB, of the substance of the larger sphere, also parallel to OP.

Now, if r and R are the radii of the smaller and larger spheres, it is obvious that $\dfrac{ds}{dS}=\dfrac{r^2}{R^2}$.

Consider the force-intensity at P due to the smaller, and at p due to the larger, sphere. Each acts in the line PO; hence to find the resultant force at P we may consider only the component attraction parallel to PO due to the bar ab and to all the other parallel bars into which the smaller sphere can be broken up. If the law of attraction is expressed by $\lambda f'(r)$, as in Art. 317, and if dX' is the intensity of attraction of the bar ab at P, we have by equation (6), Art. 317,

$$dX' = \lambda k\rho[f(Pa)-f(Pb)].$$

Similarly, if dX is the intensity of attraction at p due to the bar AB,

$$dX = \lambda K\rho\,[f(pA)-f(pB)],$$

k and K being the areas of the normal sections of the bars.

Now $Pa = pA$; $Pb = pB$; and $\dfrac{k}{K}=\dfrac{ds}{dS}=\dfrac{r^2}{R^2}$; therefore

$$\frac{dX'}{dX}=\frac{r^2}{R^2};$$

$$\therefore\quad X'=\frac{r^2}{R^2}X,$$

where X' and X are the resultant intensities of attraction at P and p due, respectively, to the smaller and larger spheres.

2. From the last result deduce a proof of the theorem that the only law of attraction for which a uniform spherical shell will exercise no resultant force at any internal point is the law of the inverse square. [This application is due to Duhamel.]

If a shell produces no attraction inside it, all the portion of the larger sphere between the two spheres may be neglected in finding the attraction of the larger at p. Hence, however great R may be, X is constant at p, so that $X' \propto \frac{1}{R^2}$, however small the inner sphere may be.

3. Calculate the intensity of attraction of a uniform thin rectangular plate at a point on the perpendicular to its plane drawn at its centre.

Let $2a$ and $2b$ be the lengths of its sides; h the height of the attracted particle, P, above O, the centre of the plate; ρ and τ the density and thickness of the plate. Break up the plate into bars parallel to the side $2a$; let y be the distance of one of these bars from O. Then the area of the normal section of this bar is τdy, and if the extremities of the bar are A and B and its middle point Q, we have for its attraction-intensity at P the expression (Art. 317)

$$2\frac{\gamma\rho\tau}{PQ}\sin APQ\,.\,dy.$$

Let $\theta = \angle QPO$; then $y = h\tan\theta$, $PQ = h\sec\theta$, and this expression becomes $2\gamma\rho\tau\alpha\dfrac{\sec\theta d\theta}{\sqrt{a^2+h^2\sec^2\theta}}$; and since the resultant attraction is along PO, we multiply this expression by $\cos\theta$. Thus we have

$$R = 4\gamma\rho\tau\alpha\int_0^{\alpha}\frac{\cos\theta d\theta}{\sqrt{h^2+a^2\cos^2\theta}},$$

where α is the extreme value of θ, i.e. $\tan^{-1}\frac{b}{h}$. Thus

$$R = 4\gamma\rho\tau\sin^{-1}\frac{ab}{\sqrt{(h^2+a^2)(h^2+b^2)}}.$$

If the plate is of infinite length ($a = \infty$),

$$R = 4\gamma\rho\tau\alpha.$$

4. Given the whole mass of a solid, find its shape so that its attraction, in any direction, on a particle placed at a given point may be a maximum. (*Solid of maximum attraction.*)

It is clear that the surface of the solid must pass through the given point, O. Let OA be the given direction, and let P and Q be any two points on the bounding surface of the solid. Consider an element of mass, dm at P, and an equal element at Q. Then, whatever

be the law of attraction, the element dm at P and the element dm at Q must give the same component attractions on O along OA; for if that of Q were the greater, advantage would be gained by transferring the element dm from P to Q.

Hence, if the law of attraction is expressed by $\phi(r)$, and if $\theta = \angle POA$, made with OA by the radius vector from O to any point on the bounding surface, we must have

$$\phi(r) . \cos\theta = \text{const.} \tag{1}$$

for all points on this surface. Hence the surface is one of revolution obtained by causing the curve (1) to revolve round OA. If $\phi(r) = \frac{\gamma}{r^2}$, the revolving curve is

$$\frac{\cos\theta}{r^2} = \frac{1}{a^2} = \text{const.} \tag{2}$$

Hence, if R is the resultant intensity of attraction along OA,

$$R = \gamma\rho \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^r \sin\theta \cos\theta \, dr \, d\phi \, d\theta$$

$$= 2\pi a\gamma\rho \int_0^{\frac{\pi}{2}} \cos^{\frac{3}{2}}\theta \sin\theta \, d\theta$$

$$= \frac{4}{5}\pi a \gamma\rho.$$

The value of a must be found from the given mass of the solid, M; and we easily find $M = \frac{4}{15}\pi\rho a^3$;

$$\therefore \quad R = \left[\frac{48\pi^2\rho^2 M}{25}\right]^{\frac{1}{3}} . \gamma.$$

The attraction-intensity of a sphere of mass M at a point on its surface would be $\left[\frac{16\pi^2\rho^2 M}{9}\right]^{\frac{1}{3}} . \gamma$; so that the former exceeds the latter in the ratio $(27)^{\frac{1}{3}} : (25)^{\frac{1}{3}}$.

The curve (2) which generates the solid by revolution round OA may be thus drawn. Describe a circle with O as centre and OA as radius; describe another circle with OA as diameter; draw any line, OMN, meeting the second circle in M and the first in N; then take OP, a mean proportional between OM and ON, and we have a point P on the required curve.

5. To find the attraction-intensity of an infinite homogeneous elliptic cylinder at any external point situated on the major axis of a transverse section.

Let C be the centre of the ellipse which is the transverse section of the cylinder through the point O at which the intensity of attraction is to be found, O lying on the major axis of the ellipse

at a distance ξ from C. Let P be any point on the circumference of the ellipse; with O as centre and OP ($=r$) as radius describe a circular arc cutting the ellipse again in P'; take a point Q on the ellipse indefinitely close to P, and with O as centre and OQ ($=r+dr$) as radius describe another circular arc cutting the ellipse again in Q'. From all points on PP' and QQ' draw lines of infinite length perpendicular to the plane of the figure, and we shall have a thin cylindrical plate of infinite length cut off from the given cylinder.

It is very easy to prove that the attraction of this plate on a unit mass at O, in the direction OC, is

$$4\gamma\rho\sin\theta\,dr,$$

where $\theta = \angle POC$, $\gamma =$ gravitation constant, $\rho =$ density of cylinder. (Consider this plate as formed of a number of bars.) Hence the attraction-intensity at O due to the whole cylinder is

$$4\gamma\rho\int\sin\theta\,dr.$$

But $\int\sin\theta\,dr = -\int r\cos\theta\,d\theta$, the other portion vanishing at both limits, since $\sin\theta = 0$ both at the beginning and end of the integration. Now if a and b are the semiaxes of the ellipse,

$$b^2(r\cos\theta-\xi)^2+a^2r^2\sin^2\theta = a^2b^2;$$

$$\therefore\quad r = b\frac{b\xi\cos\theta \pm a\sqrt{b^2\cos^2\theta-(\xi^2-a^2)\sin^2\theta}}{b^2\cos^2\theta+a^2\sin^2\theta}.$$

If we denote the values of r by r_2 and r_1, the integration will obviously contain the terms $-r_1\cos\theta\,d\theta$ and $r_2\cos\theta\,d\theta$, since after the radius vector OP passes the position of the tangent from O, the element $d\theta$ changes sign. Hence, if $-X$ is the intensity of attraction towards C,

$$X = -8\gamma\rho ab\int\frac{\sqrt{b^2\cos^2\theta-(\xi^2-a^2)\sin^2\theta}}{b^2\cos^2\theta+a^2\sin^2\theta}\cos\theta\,d\theta,$$

the limits of θ being 0 and the value for which $r_1 = r_2$, i.e. $\tan^{-1}\frac{b}{\sqrt{\xi^2-a^2}}$. Putting $\sqrt{\xi^2-c^2}\sin\theta = b\sin\phi$, we have

$$X = -8\gamma\rho ab\sqrt{\xi^2-c^2}\int_0^{\frac{\pi}{2}}\frac{\cos^2\phi\,d\phi}{\xi^2-c^2\cos^2\phi}$$

$$= -4\pi\gamma\rho\frac{ab}{c^2}(\xi-\sqrt{\xi^2-c^2}). \qquad (\alpha)$$

When the cylinder is circular, the value of this expression is easily found to be $-2\pi\gamma\rho\frac{a^2}{\xi}$.

6. Draw a diagram representing the weight of a particle in its different positions as it is brought from the centre of the earth out through its surface and to infinity.

7. What should be the masses of two small equal homogeneous spheres so that when placed with a distance of 1 centimetre between their centres their mutual attraction shall be 1 dyne?

Ans. The mass of each must be $100\sqrt{1543}$, or 3928, grammes.

8. Prove that if there be two homogeneous solids of equal density bounded by similar surfaces, their attraction-intensities, for the law of inverse square, at two points similarly situated with respect to them are in the ratio of the corresponding linear dimensions of the solids. (Newton, Prop. 72, Cor. 3.)

Hence the attraction at any point on a given diameter inside a solid homogeneous ellipsoid varies as the distance of the point from the centre.

9. If the intensity of attraction of any body at a point is vastly greater when the point is very close to the surface of the body than when it is distant from this surface by a small interval, the attraction takes place according to a law more rapid than that of the inverse square. (Newton, Prop. 72.)

10. Find the intensity of the attraction, for the law of inverse square, of any portion of a thin uniform spherical shell, cut off by a plane, at any point on its axis.

Ans. Let O be the centre of the sphere; OA the axis of the given segment, A being on the surface; AB the circular arc whose revolution round OA generates the given segment; P the position of the attracted particle on AO; $a =$ radius of sphere, $PO = c$, and β the angle PBO. Then the attraction is

$$\frac{2\pi a^2 \rho\gamma\tau}{c^2}(1-\cos\beta).$$

If AB is a semicircle and P internal, $\beta = 0$; if P is external, $\beta = \pi$.

11. If P coincides with O, find the attraction.

Ans. $\pi\rho\gamma\tau\sin^2\alpha$, where $\alpha = \angle BOA$.

12. Find the intensity of attraction of a uniform right cone at the middle point of its base.

Ans. $2\pi\gamma\rho h\sin\alpha\left[\sin\alpha+\cos\alpha-\sin\alpha\cos\alpha\left\{1+\log_e\cot\frac{\alpha}{2}\cot\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right\}\right]$,
where h and α are the height and semivertical angle of the cone.

13. A platinum wire of uniform diameter 1 mm. and 1 metre long attracts a gramme mass condensed into a point distant 1 cm. from the bar on a perpendicular to the bar at its middle point; find the magnitude of the force of attraction (specific gravity of platinum $= 22{\cdot}06$).

Ans. $\dfrac{1}{89075\times 10^3}$ dynes, nearly.

14. If the law of attraction is expressed by any function, $\phi'(r)$, of the distance, prove that the intensity of attraction of any homo-

geneous solid, estimated in a given direction, at any point P is expressed by the surface-integral

$$\int \phi(r) \cos\lambda \, dS,$$

where r is the distance from P of any point on the surface bounding the solid, dS is the element of surface area, and λ the angle made by the normal at this point with the given direction.

Take P as origin and the given direction as axis of x; at any point (x, y, z) in the mass let the element of volume $dx dy dz$ be taken, and let the attraction of this element be $\phi'(r) dx dy dz$. The component of this parallel to the axis of x is

$$\phi'(r)\frac{x}{r}\, dx dy dz, \quad \text{or} \quad \phi'(r)\frac{dr}{dx}.dx dy dz.$$

Integrating this, considering y and z constant, i.e. along a thin bar parallel to the axis of x, we have

$$[\phi(r_2)-\phi(r_1)]dy dz,$$

where r_1 and r_2 are the distances from P of the points in which this bar cuts the bounding surface. Now

$$dy dz = dS_2.\cos\lambda_2 = -dS_1.\cos\lambda_1,$$

the normal being at each point drawn outward; therefore, &c.

15. Calculate the attraction-intensity of a uniform elliptic plate at any point on the axis through its centre perpendicular to its plane.

Ans. If a, b are the semi-axes of the plate, $c = \sqrt{a^2-b^2}$, z = distance of attracted particle from centre, τ = thickness of plate, $k^2 = \frac{c^2}{a^2+z^2}$, $n = \frac{a^2}{a^2+z^2}$, the attraction-intensity is

$$\frac{4\gamma\rho\tau b.z}{a\sqrt{a^2+z^2}}\{\Pi(-n, k)-F(k)\}, \tag{1}$$

where $\Pi(-n, k)$ and $F(k)$ are the complete elliptic functions of the third and first kinds for the modulus k and parameter $-n$.

Again, this can be expressed entirely in terms of functions of the first and second kind, since the *complete* function of the third kind can be so expressed. Thus in general

$$\Pi(-n, k)-F(k) = \frac{\Delta(k', \beta)}{k'^2\sin\beta\cos\beta}\left\{\frac{\pi}{2}-E(k', \beta).F(k)\right.$$
$$\left.-[E(k)-F(k)]F(k', \beta)\right\},$$

where $k' = \sqrt{1-k^2}$, and $\sin\beta = \frac{\sqrt{1-n}}{k'}$. Hence (1) becomes

$$4\gamma\rho\tau\left\{\frac{\pi}{2}-E(k', \beta).F(k)-[E(k)-F(k)]F(k', \beta)\right\}, \tag{2}$$

where $\sin\beta = \frac{z}{\sqrt{z^2+b^2}}$.

This obviously verifies for a circular plate.

Section II.—*Theory of Potential.*

326.] **Potential due to any Attracting Mass.** Consider an element, dm, of mass occupying any point, M, and let a unit mass condensed into an infinitely small volume be brought by any agent along any path whatever, plane or tortuous, from a position P_0 to a position P; it is required to calculate the amount of work done in this passage of the unit mass by the force exerted on it by the fixed particle dm. Suppose the law of attraction to be that of the inverse square, and at any point of the path of P let r be its distance from M. In this position let the force be $\frac{\gamma dm}{r^2}$. Then for any small displacement of P—say from P to P'—along its path the work done by the attracting force is $-\frac{\gamma dm}{r^2}dr$, where dr is $MP' - MP$. Hence the work done by the attraction from P_0 to P is $-\gamma dm\int_{r_0}^{r}\frac{dr}{r^2}$ (where $MP_0 = r_0$), i. e.

$$\left(\frac{1}{r}-\frac{1}{r_0}\right)\gamma dm. \tag{1}$$

If r and r_0 are measured in centimetres, dm in grammes, and if γ is the constant of gravitation (Art. 321), this expression for the work done is in *ergs*.

Now if the field of attraction is produced by several particles dm, dm', dm'', ... at M, M', M'', ... the sum of the works done by the attractions of all these on the unit mass in the passage of the latter from any initial position P_0 to any final one, P, is

$$\left(\frac{1}{r}-\frac{1}{r_0}\right)\gamma dm + \left(\frac{1}{r'}-\frac{1}{r_0'}\right)\gamma dm' + \left(\frac{1}{r''}-\frac{1}{r_0''}\right)\gamma dm'' + \ldots, \tag{2}$$

$$\text{or } \gamma\left(\frac{dm}{r}+\frac{dm'}{r'}+\frac{dm''}{r''}+\ldots\right) - \gamma\left(\frac{dm}{r_0}+\frac{dm'}{r_0'}+\frac{dm''}{r_0''}+\ldots\right), \tag{3}$$

where r, r', r'', ... are the distances of the final position P from the several particles, and r_0, r_0', r_0'', ... the distances of the initial position from them.

If the initial position is infinitely distant from every attracting particle, $\frac{1}{r_0} = \frac{1}{r_0'} = \ldots = 0$, so that the work becomes

$$\gamma\left(\frac{dm}{r} + \frac{dm'}{r'} + \frac{dm''}{r''} + \ldots\right). \tag{4}$$

The amount of work done in bringing a particle of unit mass and infinitely small volume from any position in which the attractions exerted by the particles of any given system are zero (or insensible) to any point P in their field of attraction is called the **Potential** *of the field at that point.*

It will be seen that since the work done involves merely distances of P from the several particles, *it is wholly independent of the shape and length of the path along which P has been brought;* in other words, the attractions exerted by the several particles in the field are a system of conservative forces (Art. 272).

In the above formal definition of the Potential at any point produced by a given mass system, instead of saying that the unit particle is brought from *infinity* up to the final position P, we have said that it is to be brought from a position in which the attractive forces of the mass system are zero, although, *in general*, a position at infinity would satisfy this description. It will be shown soon, however, that there are cases in which the estimation of the work done on the unit particle *from infinity* up to the finite position P leads to infinite constants in the integration. If we define the Potential at P as the amount of work done in bringing the particle from infinity to this point, we must add the proviso that *when the particle is at infinity it is also infinitely distant from every attracting particle of the mass system*—i. e. that none of the attracting mass is contemplated as at infinity.

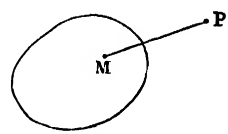

Fig. 279.

Throughout the sequel we shall speak of the position in which the forces of the field are insensible as the *zero position.*

Suppose now that the attracting particles form a continuous body of any shape represented in Fig. 279. Then the number of terms in (4) becomes infinitely great, and if we denote by V the Potential at P, we have

$$V = \gamma\int\frac{dm}{r}, \tag{α}$$

where dm is the element of mass at any point, M, and r is its distance from P. The integration is, of course, to be extended throughout the whole body, the position of P being fixed.

Thus to each position of P belongs a value, V, of the Potential. If P' is any other point at which the Potential is V', the work done by the attractions in transferring the unit particle *along any path whatever* from P' to P is

$$V - V',$$

since the particle might be brought from the zero position to P by passing through P' on the way.

It is to be remembered, then, that the expression (α) does not represent the work done in bringing a unit mass from infinity to P if any of the attracting matter is contemplated as being at infinity.

We might take $\gamma = 1$ by departing, to some extent, from the C. G. S. system, i. e. by taking the unit mass to be that which, condensed into a small sphere, attracts an equal spherical mass with a force of 1 dyne when the distance between the centres of the spheres is 1 centimetre; and this mass would be, by Ex. 7, p. 166, about 3928 grammes. We prefer, however, to adopt the C. G. S. system pure and simple and to retain γ, its value being that given in Art. 321.

It is to be observed that Potential is an *undirected* or *scalar* magnitude—unlike force, which has direction and is a *vector*. The Potential at P has magnitude but no direction.

Again, Potential is arithmetically additive; i. e. if V is the Potential at P due to any one mass system, and U the Potential at P due to any other mass system, the Potential at P due to their combined action is simply $V + U$.

327.] **Equipotential Surfaces.** The Potential produced at a point P by the attraction of any fixed masses may evidently be expressed as a function of the position of P, i. e. as a function of its co-ordinates, x, y, z, with reference to any fixed axes. If, then, $V = \phi\,(x, y, z)$, there must be a surface locus of points at each of which V has a given constant value, C; for the equation

$$\phi\,(x, y, z) = C$$

denotes a surface.

Let APB (Fig. 280) represent the surface at every point of which the Potential has the same value as that at P. [In the figure this surface is represented as closed; but, except for very

simple arrangements of attracting matter, the equipotential surfaces are very complicated, each consisting, perhaps, of several detached portions closed or unclosed.] Then no work, on the whole, is done in transferring a particle from any point P on this surface to any other point, A, on the same surface; the attractive forces of the field do as much positive work throughout a portion of any path connecting P with A as negative throughout the remainder.

If the particle is transferred from P to A *along any path lying on the equipotential surface*, then at no instant during the passage are the forces doing any work whatever; for no work is done in the passage from any point to the next consecutive.

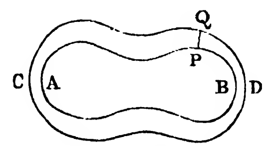

Fig. 280.

Hence the resultant attraction at any point on the surface acts along the normal to the surface at the point; for, every direction of displacement for which no work is done must be at right angles to the direction of the resultant force, and no work is done by the resultant attraction at P for any displacement of a particle at P in the tangent plane to the equipotential surface at this point.

An equipotential surface is often called a *level surface* (*surface de niveau*) from its analogy with a horizontal plane which is an equipotential surface for the case of gravity. (In reality, the equipotential surfaces for the earth's attraction are approximately spheres concentric with the earth, but a limited portion of one of them at any place may be considered a horizontal plane.) The horizontal plane is such that the work done by the weight of a particle in the descent of the particle, along any path, to the ground is the same from whatever point on the plane the particle falls; and, moreover, the particle, if placed on a smooth hard substance coinciding with this plane, would not move along it. All points on this plane have, therefore, the same Potential with reference to the earth's attraction, and are said to be at the same level. Hence the use of the term *level surface* in general, in any field of attraction, gravitational, electrostatic, or magnetic.

328.] **Relation between Force and Potential.** At any point, P (Fig. 280), construct the equipotential surface PAB; let PQ be an infinitesimal length measured on the normal at P;

and through Q describe another equipotential surface, QCD. Let V be the value of the Potential at P, and $V + \Delta V$ its value at Q. Now the resultant force at P acts along PQ, either inwards or outwards. Let it be R, and consider the work done in transferring a unit mass from P to Q. By definition this work $= \Delta V$, and if R acts from P to Q, it must also be $R \times PQ$, assuming that we may consider R as constant at all points between P and Q. Hence

$$R = \frac{\Delta V}{PQ}, \text{ or } = \frac{\Delta V}{\Delta n},$$

so that if ΔV is a positive increase of Potential, the sense of R is from P to Q. Similarly at B the magnitude of the force $= \frac{\Delta V}{BD}$, where BD is the normal distance between the two surfaces at B. Hence at different points on the same level surface the magnitude of the resultant force is inversely proportional to the normal distance between that surface and another level surface whose Potential exceeds that of the given one by an infinitesimal amount. An inspection of the figure (Fig. 280) shows the points at which the resultant force is most intense, and also those at which it is least; it is most intense where the two surfaces are closest together, and least where they are farthest apart. The value of R without approximation is to be found by diminishing PQ, or Δn, and therefore ΔV, indefinitely; i. e.

$$R = \frac{dV}{dn}, \tag{α}$$

which asserts that *at any point, P, the resultant force is the rate of increase of Potential along the normal to the level surface through the point, and it acts in the sense in which the Potential increases.*

Again, the component of force in any direction at any point, P, is the rate of variation of the Potential in that direction at P. For at P draw PP' in the given direction, meeting in P' the indefinitely close equipotential surface on which the Potential is $V + \Delta V$. Then if F is the component force along PP', and R the resultant force at P,

$$\begin{aligned} F &= R \cos QPP' \\ &= \frac{\Delta V}{\Delta n} \cos QPP' = \frac{\Delta V}{PP'}. \end{aligned}$$

Hence if $PP' = \Delta s$, and its length is diminished indefinitely,

$$F = \frac{dV}{ds}. \tag{β}$$

If ds lies anywhere in the tangent plane, the component force is zero; and the resultant force acts in the direction in which the Potential increases most rapidly.

COR. The components of force at P parallel to three fixed rectangular axes are

$$\frac{dV}{dx}, \quad \frac{dV}{dy}, \quad \frac{dV}{dz}, \tag{γ}$$

(x, y, z) being the co-ordinates of P, and V being expressed in the form $V = \phi\,(x, y, z)$.

If V is expressed as a function of the polar co-ordinates (r, θ, ϕ) of P, with reference to any origin, O, and axes, the component force along the radius vector OP is

$$\frac{dV}{dr}; \tag{δ}$$

and the component along the tangent to the parallel of latitude at P is

$$\frac{1}{r\sin\theta}\cdot\frac{dV}{d\phi}, \tag{ϵ}$$

since PP' for this direction $= r\sin\theta\,.\,\Delta\phi$; while the component along the tangent to the meridian at P is

$$\frac{1}{r}\frac{dV}{d\theta}. \tag{ζ}$$

In general, V may be expressed in terms of any three independent variables which serve as co-ordinates to define the position of a point.

Starting with the notion of work, we have deduced the force-component in any direction from the Potential. In particular, we have proved that $X = \frac{dV}{dx}$. But we might have adopted the reverse process and shown that X is the differential coefficient with respect to x of a certain function of x, y, z.

Thus (Art. 325), if $\phi\,(r) = \frac{\gamma}{r^2}$, we have

$$X = -\gamma\int\frac{x - x'}{r^3}\,dm,$$

in which the integration has reference to x', y', z'; so that we can write this in the form

$$X = \gamma \int \frac{d\left(\frac{1}{r}\right)}{dx} \cdot dm$$

$$= \frac{d}{dx}\left[\gamma \int \frac{dm}{r}\right] = \frac{dV}{dx},$$

if we denote $\gamma \int \frac{dm}{r}$ by V.

For any law of attraction, $\phi'(r)$, between elements of mass, the value of X is (Art. 325) equal to $-\int \phi'(r) \frac{x-x'}{r} dm$, or $-\int \frac{d\phi(r)}{dx} dm$, or $\frac{dV}{dx}$ if we denote $-\int \phi(r)\, dm$ by V.

Now $-\int \phi(r) dm$ is precisely the work done by the attraction on a unit mass from a zero position to the point P considered. For, the attraction exerted by dm at any distance being $\phi'(r)\, dm$, the element of work done by this for a small displacement of P is $-\phi'(r) dm \,.\, dr$, and the whole amount done from the zero position is $-dm \int \phi'(r) dr$, or $-\phi(r) dm$. Summing the works done by all the other elements of attracting mass, we have

$$V = -\int \phi(r)\, dm. \tag{η}$$

The process, however, of deducing the idea and properties of Potential from the components of force is less in accordance with the methods of modern Physics than the reverse process, which we have here adopted.

It will be useful to the student to imagine the whole field of attraction, due to any arrangement of mass, as mapped out by a series of equipotential surfaces, the value of the Potential increasing from one surface to the next by a small constant amount.

329.] **Differential Equations of Potential.** At any point P describe the usual small rectangular parallelepiped whose edges are parallel to the axes of x, y, z. If in Fig. 228 we put P in place of O, and take the edges infinitely small, equal to dx, dy, dz, the parallelepiped there represented is such as we contemplate. Now take the surface-integral of normal force-intensity over this parallelepiped. The *outward* normal force-

intensity on the face $PBFC$ is $-X$ or $-\dfrac{dV}{dx}$; so that the contribution of this face is $-\dfrac{dV}{dx}dydz$; while the contribution of the opposite face is

$$\frac{dV}{dx}dydz + \frac{d}{dx}\left(\frac{dV}{dx}dydz\right).dx;$$

hence the sum contributed by these two faces is $\dfrac{d^2V}{dx^2}dxdydz$. Similarly the sum contributed by the two faces perpendicular to the axis of y is $\dfrac{d^2V}{dy^2}dxdydz$, and that contributed by the remaining faces is $\dfrac{d^2V}{dz^2}dxdydz$. The whole surface-integral for the elementary volume considered is therefore

$$\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)dxdydz,$$

or $\nabla^2V.dxdydz$, using the symbol

$$\nabla^2 \text{ for } \frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}.$$

Now if there is none of the attracting matter within the element of volume at P, this quantity must be zero, by Art. 324. Hence at every point in space at which none of the attracting matter exists

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0, \text{ or } \nabla^2V=0. \qquad (\alpha)$$

If, on the contrary, P is a point inside the attracting matter, and if ρ is the density, or mass per unit volume (cubic centimetre) at P, the mass contained in the parallelepiped is $\rho\,dxdydz$; so that by Art. 324,

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=-4\pi\gamma\rho, \text{ or } \nabla^2V=-4\pi\gamma\rho.... \qquad (\beta)$$

Equation (α) is known as *Laplace's Equation*, while (β) is *Poisson's Equation.*

We now proceed to find the equivalent equations in polar co-ordinates. To do this, we take the surface-integral of normal force-intensity over the polar element of volume *msqt* (Fig. 219, vol. i). Let s in this figure represent the point, P, in any

field of attraction, and let the co-ordinates of s be (r, θ, ϕ), let the normal force-intensity on the face msq, measured in the sense Os, be R, while the area of this face $= s_1$. Then this face will contribute the term $-Rs_1$ to the surface-integral, while the opposite face will contribute $Rs_1 + \frac{d(Rs_1)}{dr} dr$; therefore these faces give conjointly $\frac{d(Rs_1)}{dr} dr$. Let the normal force-intensities on the faces mst and tsq be T and S, and the areas of these faces s_2 and s_3; then the first and its opposite face will conjointly give $\frac{d(Ts_2)}{d\theta} d\theta$; and the second with its opposite will give $\frac{d(Ss_3)}{d\phi} d\phi$. Hence

$$\frac{d(Rs_1)}{dr} dr + \frac{d(Ts_2)}{d\theta} d\theta + \frac{d(Ss_3)}{d\phi} d\phi = 0, \text{ or}$$
$$= -4\pi\gamma\rho r^2 \sin\theta \, dr \, d\theta \, d\phi,$$

according as there is not, or is, mass inside the element of volume.

Now $\quad R = \frac{dV}{dr}, \quad T = \frac{1}{r}\frac{dV}{d\theta}, \quad S = \frac{1}{r\sin\theta}\frac{dV}{d\phi};$

$$s_1 = r^2 \sin\theta \, d\theta \, d\phi, \quad s_2 = r\sin\theta \, dr \, d\phi, \quad s_3 = r \, d\theta \, dr,$$

so that the equations are

$$\frac{1}{r^2}\left[\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2V}{d\phi^2}\right] = 0,$$
$$\text{or } -4\pi\gamma\rho; \quad (\gamma)$$

and it will be useful to note the identity (putting μ for $\cos\theta$)

$$\nabla^2 V \equiv \frac{1}{r^2}\left[\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) + \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dV}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2V}{d\phi^2}\right]. \quad (\delta)$$

A result of importance may here be noted—namely, if the equation $\nabla^2 V = 0$ is satisfied by the value $V = r^n Y$, where Y is a function of θ and ϕ only, it will also be satisfied by the value $V = \frac{Y}{r^{n+1}}$; for, each of these values when substituted in (γ) gives the equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dY}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2Y}{d\phi^2} + n(n+1) Y = 0.$$

Equation for V in Cylindrical Co-ordinates. The position of

any point, P, in space may be defined in the following manner by what are called *cylindrical co-ordinates*. Take any fixed rectangular co-ordinate axes, Ox, Oy, Oz; from P draw PM perpendicular to the plane of xy, meeting this plane in M. Then the cylindrical co-ordinates of P are the lengths PM and OM, and the angle MOx. Denote these, respectively, by (z, ζ, ϕ); then V at P must be expressible as a function of these. The corresponding small element of volume at P is obtained by drawing a cylinder passing through P having Oz for axis, and another cylinder very close to it (having for radius $\zeta + d\zeta$); a plane through P parallel to the plane xy, and another plane parallel to this at a distance dz from it; an 'azimuth plane', PMO, containing P and Oz, and finally a close azimuth plane through Oz making the angle $d\phi$ with the previous azimuth plane. The volume of this element is $\zeta dz\, d\zeta d\phi$, and the areas s_1, s_2, s_3 of its faces through P are $s_1 = \zeta d\,\zeta d\phi$, $s_2 = \zeta dz\, d\phi$, $s_3 = dz\, d\zeta$; and the force-intensity perpendicular to the first and measured *outwards* from the surface of the element of volume is $-\frac{dV}{dz}$, so that this face gives $-s_1\frac{dV}{dz}$, and its opposite gives

$$s_1\frac{dV}{dz} + \frac{d}{dz}\left(s_1\frac{dV}{dz}\right)\cdot dz$$

to the surface-integral. The sum of these is

$$\frac{d}{dz}\left(s_1\frac{dV}{dz}\right)\cdot dz.$$

Similarly the other pairs of opposite faces contribute

$$\frac{d}{d\zeta}\left(s_2\frac{dV}{d\zeta}\right)\cdot d\zeta \text{ and } \frac{d}{d\phi}\left(\frac{s_3}{\zeta}\frac{dV}{d\phi}\right)\cdot d\phi,$$

so that the whole surface-integral over this element of volume is

$$\left[\zeta\frac{d^2V}{dz^2} + \frac{d}{d\zeta}\left(\zeta\frac{dV}{d\zeta}\right) + \frac{1}{\zeta}\frac{d^2V}{d\phi^2}\right] dz\, d\zeta d\phi.$$

Hence the equations for V are

$$\frac{d^2V}{dz^2} + \frac{1}{\zeta}\frac{d}{d\zeta}\left(\zeta\frac{dV}{d\zeta}\right) + \frac{1}{\zeta^2}\frac{d^2V}{d\phi^2} = 0, \text{ or } = -4\pi\gamma\rho; \qquad (\epsilon)$$

and we have the identity

$$\nabla^2 V \equiv \frac{d^2V}{dz^2} + \frac{d^2V}{d\zeta^2} + \frac{1}{\zeta}\frac{dV}{d\zeta} + \frac{1}{\zeta^2}\frac{d^2V}{d\phi^2}. \qquad (\zeta)$$

If the attracting matter is symmetrical, as to shape and density, about an axis (that of z, suppose), equations (ϵ) and (γ) become

$$\frac{d^2V}{dz^2}+\frac{1}{\zeta}\frac{d}{d\zeta}\left(\zeta\frac{dV}{d\zeta}\right)=0,\ \text{or}=-4\pi\gamma\rho,$$

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right)+\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dV}{d\mu}\right\}=0,\ \text{or}=-4\pi\gamma\rho r^2,$$

and these are necessarily the same, and can be transformed one into the other by the relations $r=\sqrt{z^2+\zeta^2}$, $\theta=\tan^{-1}\frac{\zeta}{z}$, which give

$$\frac{d}{dz}=\cos\theta\frac{d}{dr}-\frac{\sin\theta}{r}\frac{d}{d\theta},$$

$$\frac{d}{d\zeta}=\sin\theta\frac{d}{dr}+\frac{\cos\theta}{r}\cdot\frac{d}{d\theta}.$$

330.] **Infinite Elliptic Cylinder.** In general, *to find the Potential at any point due to an infinite homogeneous cylinder whose transverse section is any plane curve symmetrical with respect to an axis, it is sufficient to know the value of the Potential at all points on this axis.* (Laplace, *Mécanique Céleste*, Vol. I, Book III, Chap. 6.)

For, if the axis of z is taken parallel to the axis of the cylinder, V will be a function of x and y only, and the equation for V will be

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}=0.$$

The solution of this partial differential equation is

$$V=F(x+y\sqrt{-1})+f(x-y\sqrt{-1}),$$

where F and f are two arbitrary functions.

Let the axis of x be taken coincident with the axis of symmetry of the transverse section; then the above value of V must be unaltered if for x and y we put x and $-y$, since V is obviously the same at the point $(x, -y)$ as at the point (x, y).

$$\therefore\quad V=f(x+y\sqrt{-1})+F(x-y\sqrt{-1}).$$

Hence $$2V=(F+f)\,(x+y\sqrt{-1})+(F+f)\,(x-y\sqrt{-1}),$$

$$=\phi\,(x+y\sqrt{-1})+\phi\,(x-y\sqrt{-1}),\qquad(\alpha)$$

so that at every point on the axis, if U is the Potential,

$$U=\phi\,(x),$$

and, by hypothesis, this is known, i. e. the form of the function ϕ is known. Then if in ϕ we put $x+y\sqrt{-1}$ and $x-y\sqrt{-1}$ for x successively and add the results, we get $2V$, by (α).

Similarly for the attraction-intensity. Its value at any point on the axis of symmetry of the transverse section is $\phi'(x)$, while if X and Y are its components at any point,

$$2X = \phi'(x+y\sqrt{-1})+\phi'(x-y\sqrt{-1}), \qquad (\beta)$$

$$2Y = \sqrt{-1}\,[\phi'(x+y\sqrt{-1})-\phi'(x-y\sqrt{-1})], \qquad (\gamma)$$

which are both known when ϕ' is known.

To apply this to the case of an infinite elliptic cylinder, the form of ϕ' has been already found (Example 5, Art. 325). Hence we have for the attraction-intensity at any point (x, y),

$$-2X = 4\pi\gamma\rho\frac{ab}{c^2}\left[x+y\sqrt{-1}-\sqrt{(x-y\sqrt{-1})^2-c^2}\right]$$
$$+4\pi\gamma\rho\frac{ab}{c^2}\left[x-y\sqrt{-1}-\sqrt{(x-y\sqrt{-1})^2-c^2}\right],$$

$$-2Y = 4\pi\gamma\rho\frac{ab}{c^2}\sqrt{-1}\left[x+y\sqrt{-1}-\sqrt{(x+y\sqrt{-1})^2-c^2}\right]$$
$$-4\pi\gamma\rho\frac{ab}{c^2}\sqrt{-1}\left[x-y\sqrt{-1}-\sqrt{(x-y\sqrt{-1})^2-c^2}\right];$$

or

$$X- = 2\pi\gamma\rho\frac{ab}{c^2}\Big[2x-\sqrt{x^2-y^2-c^2+2xy\sqrt{-1}}$$
$$-\sqrt{x^2-y^2-c^2-2xy\sqrt{-1}}\Big],$$

$$-Y = 2\pi\gamma\rho\frac{ab}{c^2}\sqrt{-1}\Big[2y\sqrt{-1}-\sqrt{x^2-y^2-c^2+2xy\sqrt{-1}}$$
$$+\sqrt{x^2-y^2-c^2-2xy\sqrt{-1}}\Big].$$

These may be put into real forms by observing that if

$$\sqrt{A+B\sqrt{-1}}+\sqrt{A-B\sqrt{-1}} = u,$$

we have
$$u = \sqrt{2}\sqrt{A+\sqrt{A^2+B^2}}.$$

Hence

$$-X = 2\pi\gamma\rho\frac{ab}{c^2}\left[2x-\sqrt{2}\sqrt{x^2-y^2-c^2+\sqrt{(x^2-y^2-c^2)^2+4x^2y^2}}\right], \qquad (\delta)$$

$$-Y = 2\pi\gamma\rho\frac{ab}{c^2}\left[-2y+\sqrt{2}\sqrt{\sqrt{(x^2-y^2-c^2)^2+4x^2y^2}-(x^2-y^2-c^2)}\right], \qquad (\epsilon)$$

If the point (x, y) is on the surface of the cylinder, $x = a\cos\phi$, $y = b\sin\phi$, and

$$X = -4\pi\gamma\rho\frac{ab}{a+b}\cos\phi, \qquad (\zeta)$$

$$Y = -4\pi\gamma\rho\frac{ab}{a+b}\sin\phi, \qquad (\eta)$$

so that the resultant is constant in magnitude, and it acts in a line parallel to the radius of the auxiliary circle of the ellipse.

331.] **Potential Work, or Static Energy, of a Self-Attracting System.** In a system in which forces of attraction are exerted between particle and particle, these forces will do an amount of (positive or negative) work if the form of the system is altered. We propose to find the amount of work thus done in a material system self-attracting according to the Newtonian law.

Consider a system of particles of masses $m_1, m_2, m_3, \ldots$ with distances $r_{12}, r_{13}, \ldots, r_{23}, \ldots$ between them in any given configuration, and with distances $r'_{12}, r'_{13}, \ldots, r'_{23}, \ldots$ between them in any final configuration.

First, let m_1 alone be brought into the second configuration, all the others being fixed. Then the amount of work done by the forces of attraction acting on it is

$$\gamma m_1\left[\left(\frac{1}{\rho_{12}}-\frac{1}{r_{12}}\right)m_2+\left(\frac{1}{\rho_{13}}-\frac{1}{r_{13}}\right)m_3+\ldots\right],$$

where $\rho_{12}, \rho_{13}, \ldots$ are the distances between m_1 and $m_2, m_3, \ldots$ after this change. Now let m_2 be brought into the final position, $m_3, m_4, \ldots$ being kept fixed. The amount of work thus done is

$$\gamma m_2\left[\left(\frac{1}{r'_{12}}-\frac{1}{\rho_{12}}\right)m_1+\left(\frac{1}{\rho_{23}}-\frac{1}{r_{23}}\right)m_3+\ldots\right].$$

Bringing m_3 now into the final position, $m_4, \ldots$ being fixed, the work is

$$\gamma m_3\left[\left(\frac{1}{r'_{13}}-\frac{1}{\rho_{13}}\right)m_1+\left(\frac{1}{r'_{23}}-\frac{1}{\rho_{23}}\right)m_2+\ldots\right].$$

Repeating this process for all the rest, and adding the works done, we have the whole work $\left(\text{multiplied by } \frac{1}{\gamma}\right)$,

$$= m_1 m_2\left(\frac{1}{r'_{12}}-\frac{1}{r_{12}}\right)+m_1 m_3\left(\frac{1}{r'_{13}}-\frac{1}{r_{13}}\right)+m_1 m_4\left(\frac{1}{r'_{14}}-\frac{1}{r_{14}}\right)+\ldots$$

$$+m_2 m_3\left(\frac{1}{r'_{23}}-\frac{1}{r_{23}}\right)+m_2 m_4\left(\frac{1}{r'_{24}}-\frac{1}{r_{24}}\right)+\ldots$$

$$+m_3 m_4\left(\frac{1}{r'_{34}}-\frac{1}{r_{34}}\right)+\ldots.$$

Now rearrange this by taking one-half of the first, second, third, ... terms in the first row and putting them, respectively, into the succeeding rows, and similarly treating the terms of the other rows. We thus find that the expression is the same as

$$\tfrac{1}{2}m_1\left[\frac{m_2}{r'_{12}}+\frac{m_3}{r'_{13}}+\ldots-\frac{m_2}{r_{12}}-\frac{m_3}{r_{13}}-\ldots\right]$$

$$+\tfrac{1}{2}m_2\left[\frac{m_1}{r'_{12}}+\frac{m_3}{r'_{23}}+\ldots-\frac{m_1}{r_{12}}-\frac{m_3}{r_{23}}-\ldots\right]+\&c.,$$

or
$$\tfrac{1}{2}(V_1'-V_1)m_1+\tfrac{1}{2}(V_2'-V_2)m_2+\tfrac{1}{2}(V_3'-V_3)m_3+\ldots, \tag{α}$$

all divided by γ, where V_1' is the value of the potential in the final position of m_1 and V_1 its value in the first position of m_1, with similar meanings of V_2', V_2, &c.

Or we may write the work in the form

$$\tfrac{1}{2}(\Sigma Vm)'-\tfrac{1}{2}(\Sigma Vm), \tag{β}$$

where $(\Sigma Vm)'$ means the sum obtained by multiplying the mass of each particle of the system by the value of the potential at its position in the final configuration, and ΣVm the corresponding quantity in the first configuration.

If the particles are infinitely numerous and form a continuous mass, the work of the forces of attraction in changing the configuration is

$$\tfrac{1}{2}(\int Vdm)'-\tfrac{1}{2}(\int Vdm). \tag{γ}$$

Hence to scatter the particles of a given self-attracting system to (practically) infinite distances from each other requires an amount of work equal to

$$\tfrac{1}{2}\int Vdm, \tag{δ}$$

in which expression the integral is taken throughout the system in its given configuration. This expression (δ) may, therefore, be regarded as the Potential Work of the forces of the system, or its Static Energy, in this configuration.

Again, if V_1, V_2, ... are the potentials at the positions of a number of particles m_1', m_2', ... produced by a given system of particles m_1, m_2, ..., and if the system m_1', m_2', ... (which we shall denote by M') either is not self-attractive or is absolutely

rigid, the work of removing the system M' completely out of the field of attraction of the other system (which we denote by M) is obviously

$$(m_1' V_1 + m'_2 V_2 + m'_3 V_3 + \ldots), \text{ or } \Sigma m' V;$$

or, again, $\int V dm'$, if the system M' forms a continuous mass.

But the work of removing the system M' out of the field of influence of M must be exactly the same as the work of removing M out of the field of influence of M'—since each is the work of separating the two attracting systems, each of which is considered as either rigid or not self-attractive.

But if V_1', V_2', ... be the values of the potential produced by the system M' at the positions of m_1, m_2, ... the expression for the work of removing M is

$$(m_1 V_1' + m_2 V_2' + \ldots) \text{ or } \Sigma m V',$$

or $\int V' dm$.

Hence we have a useful theorem due to Gauss, viz.

$$\int V dm' = \int V' dm. \qquad (\epsilon)$$

But this is also evidently true if the elements dm, dm' are multiplied by any function of the distance between them, as well as when this function is $\frac{1}{r}$; and, moreover, instead of two mass systems, M and M', we may have two volumes of empty space, so that if dm and dm' are elements of volume, equation (ϵ) still holds. The theorem in this case is of course not physical but merely analytical.

We shall find useful applications of this theorem of Gauss hereafter.

332.] **Magnetic Shell.** In the study of Magnetism we have to deal with a *magnetic shell*, which behaves like a material shell consisting of two layers indefinitely close together, each element of one of the layers—the outer, suppose—acting on a given material particle, placed anywhere, with a *repulsive* force following the Newtonian law, while each element of the other layer *attracts* the same particle according to the same law. Let Fig. 278, Art. 322, represent such a shell, and suppose the points P and Q to be on the outer and inner layers, respectively. The outer layer we may imagine to be composed of *positive matter*, the amount of which per unit area is m at any point P; while at Q, the point directly opposite to P, on the inner shell we may imagine a quantity of *negative matter*, equal to $-m$ per unit area.

The inner shell is, then, wholly composed of *negative matter*, and the amounts of + and − matter, per unit area, are equal at the extremities of the (small) normal distance between the shells at all points. The terms 'positive' and 'negative' matter are, of course, only provisional; they stand merely for *causes of repulsion and attraction*. Again, the quantity m may vary from point to point on either shell. The product of m and the normal distance, Δn, between the shells at any point is called the *strength* of the shell at that point. Denote this product by ϕ; so that

$$\phi = m\,\Delta n.$$

We shall assume the shell to be of constant strength at all points; so that if the surface-density, m, of matter varies along either layer, the normal distance between the layers will also vary—but in such a way that ϕ remains constant.

For ordinary gravitating matter, whose constant of gravitation has the numerical value of γ previously given, such a combination of indefinitely close layers of repulsive and attractive matter would be almost absolutely nugatory—unproductive of anything but an infinitesimal force effect at any point—since, Δn being at all points infinitesimal, the product $m\Delta n$ would be infinitely small; but if a *very large* quantity of repulsive 'matter' could be concentrated on a small surface, the product $m\Delta n$ might not be infinitesimal, and the whole action of such a shell on a unit mass might amount to a very considerable force.

The discussion of the following properties of such a shell as we now imagine will not only serve to illustrate the subject of the present Chapter but prove a useful study for the student of the theory of Magnetism.

(*a*) *The potential produced by a magnetic shell at any point in space is proportional to the conical angle subtended at the point by the bounding edge of the shell.*

Let A be the point at which the value of the potential is to be found; let Q be any point on the inner surface, and P the opposite point on the outer surface, of the shell; let $AQ = r$, $AP = r + \Delta r$. Also let the constant of gravitation for the kind of matter now supposed be k—i.e., the number of dynes in the force of repulsion between two positive unit masses at a distance of 1 cm.—; suppose a unit mass placed at A; take any small element of area, dS, of the inner layer at Q, and on the contour of this erect a cylinder which will cut off an equal element of

area, dS, on the outer at P. The quantities of matter on these elements being, respectively, $-mdS$ and mdS, the sum of their potentials at A is

$$k\left(\frac{1}{r} - \frac{1}{r+\Delta r}\right) mdS, \text{ or } k\frac{m\,\Delta r}{r^2}dS. \tag{1}$$

Now if ψ is the angle made by AP with the normal to the shell at P, we have $\Delta r = \Delta n \,.\cos\psi$, so that this element of potential becomes

$$k\phi\frac{\cos\psi}{r^2}dS, \text{ or } k\phi\,.\,d\omega, \tag{2}$$

by Art. 316, where $d\omega$ is the conical angle subtended at A by the element dS of the surface of the shell. It is usual to assume the constant k equal to unity—which amounts to taking the unit mass as indicated near the end of Art. 326. On this understanding, then, if V is the potential of the shell at A, we have

$$V = \phi\,.\,\omega, \tag{3}$$

where ω is the conical angle subtended by the whole shell at A, i. e. the conical angle subtended by its bounding edge.

Hence if the bounding edge disappears—in other words, if the shell is a closed surface—it produces a zero potential, and therefore a null force effect, at all points outside it, and also a uniform potential, $4\pi\phi$, and null force effect, at all points inside it.

Hence also all magnetic shells of the same strength which have the same bounding edge produce the same effects at all points in space.

(*b*) *The potential produced by a magnetic shell at any point in space is proportional to the normal flux of force through the surface of the shell produced by a unit particle at the point.*

This follows at once from Art. 324.

(*c*) *If a magnetic shell is placed in any field of force which has a potential satisfying Laplace's equation, the whole action of the field on the shell can be produced by a distribution of force along its bounding edge only, according to a simple law.*

Let X, Y, Z be the components of the force-intensity of the field (forces exerted on the magnetic unit) at any point. Then we assume

$$X = \frac{dU}{dx}, \quad Y = \frac{dU}{dy}, \quad Z = \frac{dU}{dz},$$

where U is the potential of the field at the point. Hence

$$\frac{dX}{dy} = \frac{dY}{dx}, \text{ \&c.}$$

Calculate now the whole x-component of force exerted on the shell. On the quantity $-mdS$ at any point, Q, on the inner layer, the force is $-mXdS$. If l, m, n are the direction-cosines of the normal at Q, ν the thickness of the shell at Q, and x, y, z the co-ordinates of Q, the co-ordinates of P are $x + l\nu, y + m\nu, z + n\nu$; so that the value of X at P is

$$X + \nu\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)X.$$

Hence the resultant x-component on the corresponding elements at P and Q is

$$\phi\int\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)X\,.\,dS;$$

and the whole x-force on the shell is

$$\phi\int\left(l\frac{dX}{dx} + m\frac{dX}{dy} + n\frac{dX}{dz}\right)dS. \tag{4}$$

Now since $\nabla^2 U = 0$, we have

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0.$$

Substituting from this the value of $\frac{dX}{dx}$, and also putting $\frac{dY}{dx}$ for $\frac{dX}{dy}$ and $\frac{dZ}{dx}$ for $\frac{dX}{dz}$ in (4), we have (4) equal to

$$\phi\int\left\{\left(n\frac{d}{dx} - l\frac{d}{dz}\right)Z - \left(l\frac{d}{dy} - m\frac{d}{dx}\right)Y\right\}dS,$$

or

$$\phi\int(\delta_2 Z - \delta_3 Y)\,dS. \tag{5}$$

But by Theorem 2, Art. 316, a, the first term in this integral is equal to the integral $\phi\int Z\frac{dy}{ds}\cdot ds$ taken along the bounding edge of the shell, while the second term is equal to the integral $-\phi\int Y\frac{dz}{ds}ds$ taken along this edge. Hence the whole x-component, F_x, of force on the shell is given by the equation

Similarly

$$\left.\begin{aligned} F_x &= \phi\int\left(Z\frac{dy}{ds} - Y\frac{dz}{ds}\right)ds\,. \\ F_y &= \phi\int\left(X\frac{dz}{ds} - Z\frac{dx}{ds}\right)ds\,, \\ F_z &= \phi\int\left(Y\frac{dx}{ds} - X\frac{dy}{ds}\right)ds\,, \end{aligned}\right\} \tag{6}$$

where F_y, F_z, are the components of force, parallel to the other axes, exerted by the field on the shell.

Now if R is the resultant force-intensity of the field at any point, P, of the bounding edge, and θ the angle between R and the tangent to the edge at P, the multipliers of ds in equations (6) are simply the x, y, and z components of a force

$$R \sin \theta \tag{7}$$

acting along the line which is at once perpendicular to R and to the tangent to the edge at P. This force, R', may be graphically represented thus: at any point, P, on the edge of the shell draw a line representing in magnitude and direction the resultant force-intensity, R, of the field of foice; draw also at P a unit length in the direction of the tangent to the edge at P, and complete the parallelogram determined by these two lines; then at P draw a perpendicular to the plane of this parallelogram proportional to its area; this perpendicular will represent the magnitude and direction of the force R' to be applied to the edge at P, per unit length. As to the sense in which the perpendicular to the plane is to be drawn, a watch-hand rule similar to that in Art. 200 may be adopted; or we may express the result by a quaternion notation thus: let a unit vector, τ, be drawn along the tangent at P to the edge in the sense in which a man walking on the positive side of the shell along the edge must travel so as to keep the shell at his left hand, and let R be the vector representing the resultant force-intensity at P; then

$$R' = V\tau R. \tag{8}$$

We have now to show that the system R' will produce the same moment about any axis as the force system (X, Y, Z).

To calculate the moment of the latter about the axis of x, let Q be a point on the inner layer and P on the outer, as before. Then the moment of force exerted on the element, $-mdS$, at Q is

$$-(Zy - Yz)\, mdS,$$

and therefore the resultant moment given by the masses $-mdS$ and mdS at Q and P is

$$\phi\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)(Zy - Yz)\,.\,dS.$$

But this is easily seen to be the same as

$$\phi\left\{\left(m\frac{d}{dz}-n\frac{d}{dy}\right)(yY+zZ)-\left(n\frac{d}{dx}-l\frac{d}{dz}\right)yX-\left(l\frac{d}{dy}-m\frac{d}{dx}\right)zX\right\}dS;$$

or $$\phi\left\{\delta_1(yY+zZ)-\delta_2(yX)-\delta_3(zX)\right\}dS,$$

with the notation of Theorem 2, Art. 316, *a*; and by this Theorem the result is the line-integral

$$\phi\int\left\{(yY+zZ)\frac{dx}{ds}-yX\frac{dy}{ds}-zX\frac{dz}{ds}\right\}ds,$$

taken along the edge of the shell. Hence if L denotes this moment,

$$L=\phi\int\left\{y\left(Y\frac{dx}{ds}-X\frac{dy}{ds}\right)-z\left(X\frac{dz}{ds}-Z\frac{dx}{ds}\right)\right\}ds. \quad (9)$$

Now the coefficient of ds is exactly the moment of the force R' about the axis; therefore the system of edge-forces, R', is completely equivalent to the given forces acting on all the elements of the shell.

(*d*) To express the Static Energy of two magnetic shells occupying given positions.

Let their strengths be ϕ and ϕ'.

Take any point, Q', on the inner (supposed negative) surface of the second shell. The potential at this point due to the first is $\phi\omega$ by (3); and if P' is the point on the outer (positive) surface at the extremity of the normal at Q', the potential at P' is

$$\phi\omega+\phi\nu'\left(l'\frac{d}{dx'}+m'\frac{d}{dy'}+n'\frac{d}{dz'}\right)\omega,$$

where ν' is the thickness of the shell, l', m', n' are the direction-cosines of the normal, and (x', y', z') the co-ordinates of Q'. Hence the potential work of the force of the first shell on the masses $-m\,dS'$ and $m\,dS'$ at Q' and P' is

$$\phi\phi'\left(l'\frac{d}{dx'}+m'\frac{d}{dy'}+n'\frac{d}{dz'}\right)\omega\,.\,dS'.$$

Now by Art. 316, *b*, this is the same as

$$\phi\phi'\left\{l'\left(\frac{dH}{dy'}-\frac{dG}{dz'}\right)+m'\left(\frac{dF}{dz'}-\frac{dH}{dx'}\right)+n'\left(\frac{dG}{dx'}-\frac{dF}{dy'}\right)\right\}dS';$$

and the integral of this over the surface of the shell is by Theorem 3, Art. 316, *a*, the line-integral

$$\phi\phi'\int\left(F\frac{dx'}{ds'}+G\frac{dy'}{ds'}+H\frac{dz'}{ds'}\right)ds',$$

taken along the edge of the shell. Substituting for F its value, $\int \frac{dx}{r}$, and similar values of G, H, the potential work of the forces of the first shell acting on the second is

$$\phi\phi' \iint \frac{1}{r}\left(\frac{dx}{ds}\frac{dx'}{ds'} + \frac{dy}{ds}\frac{dy'}{ds'} + \frac{dz}{ds}\frac{dz'}{ds'}\right) ds\, ds',$$

this double integral being taken over the edges of the two shells, (x, y, z) being the co-ordinates of any point, P, on the edge of the first, (x', y', z') those of any point, P', on the edge of the second, r being the distance PP', and ds, ds' elements of length of the edges at P and P'. If ϵ is the angle between the directions of ds and ds', and W stands for the Static Energy,

$$W = \phi\phi' \iint \frac{\cos\epsilon}{r}\, ds\, ds', \tag{10}$$

which is known as *Neumann's Formula.*

The Static Energy here expressed is merely the work which must be done against their mutual forces in withdrawing either shell, considered as a rigid body, to an infinite distance from the other. The result depends, then, merely on the shapes and positions of the *edges* and not at all on those of the *surfaces* of the shells.

(*e*) Static Energy of a Magnetic Shell and any Field of Force. Supposing the field of force to have at each point a potential, the static energy, in any position of the shell, is equal to the normal flux of force of the field through the shell, multiplied by the strength of the shell.

For, taking, as before, any points Q and P, at the extremities of the small normal thickness, on the negative and positive faces of the shell, if V is the potential of the field at Q, the potential work of the forces on the element $-m dS$ at Q is $-mV dS$, while for the element $m dS$ at P it is

$$mV dS + m\nu dS\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)V,$$

where ν is the thickness of the shell at P. Hence if W is the whole potential work

$$W = \phi\int\left(l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz}\right)dS, \tag{11}$$

which, since $\frac{dV}{dx}, \frac{dV}{dy}, \frac{dV}{dz}$ are the components of the force-inten-

sity of the field at P (or Q), is the normal flux of force-intensity of the field through the shell. When $\nabla^2 V = 0$, this can be expressed as a line-integral of the vector (u, v, w) along the edge of the shell by determining u, v, w as at the end of Art. 316, *a*.

EXAMPLES.

[Throughout these examples it may be assumed that length and mass are measured in centimetres and grammes, so that the constant of gravitation, $\gamma, = \frac{1 \text{ dyne}}{1543 \times 10^4}$; and V is in *ergs per gramme.*]

1. If the field of attraction is produced by two particles of masses m_1 and m_2 at two points N and S (Fig. 36, vol. i), and if r_1 and r_2 are the distances of any point P from them,

$$V = \gamma\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right),$$

γ being the gravitation constant (Art. 321).

Now m_1 and m_2 being both essentially positive, very large values of V will correspond to points P very near either N or S, while small values will correspond to points very distant from both, and zero values to points at infinity. The equipotential surfaces are evidently all surfaces of revolution round the line NS. If V is a very large constant, the equipotential surface will consist approximately of a sphere with centre N and radius $= \frac{\gamma m_1}{C}$ together with a sphere with centre S and radius $\frac{\gamma m_2}{C}$. As the values of V decrease, the equipotential surfaces are each formed by two oval shaped surfaces surrounding the points N and S; for a certain value of V these ovals join each other at a point between N and S, forming a surface generated by a kind of lemniscate revolving round NS; and for less values of V each surface becomes continuous, and is nearly a sphere for very distant points.

For Newtonian gravitation, however, if the masses m_1 and m_2 have moderate values—say a few grammes each—large values of V exist only at points infinitesimally distant from N or S. Thus if $m_1 = 1$ gramme and $m_2 = 2$ grammes, and if V is only 1 erg, the radius of the sphere round N is γ centimetres, i.e. $\frac{1}{1543 \times 10^4}$ cms., which is practically zero.

Unless the masses condensed at N and S are comparable with 1543×10^4 grammes, no sensible values of V (i.e. of the work of bringing 1 gramme mass from infinity into the neighbourhood of NS) will exist, except at infinitesimal distances from the points N and S.

This inconvenience does not exist in Electrostatics and Magnetism, because in these domains the analogues of the unit (gramme) mass

in Newtonian gravitation act upon each other at small distances with forces incomparably greater than that exerted by two condensed grammes at a distance of 1 cm.

If we suppose m_1 to exercise a repulsive force at P, while m_2 exerts an attractive force, we shall have

$$V = \gamma\left(-\frac{m_1}{r_1} + \frac{m_2}{r_2}\right),$$

and the surface of zero potential, instead of being wholly at infinity, is a sphere, with regard to which N and S are inverse points (Art 319).

The field produced by both particles together may be studied by superposing the fields produced by them separately. Thus the equipotential surfaces due to each are spheres. Describe round N the spheres for which the potential due to m_1 are C, $C+k$, $C+2k\ldots$ where k is any small potential magnitude; and round S the spheres for which V is C', $C'-k$, $C'-2k,\ldots$; then the curves of intersection of these trace out the surface on which the potential is $C+C'$.

2. To calculate V at any point for a thin uniform bar (see Fig. 276, Art. 317).

With the same notation as before,

$$V = \gamma k\rho\int\frac{ds}{PM} = -\gamma k\rho\int_{\pi-B}^{A}\frac{d\theta}{\sin\theta} = \gamma k\rho\log\left(\cot\frac{A}{2}\cot\frac{B}{2}\right). \quad (\alpha)$$

This may be put into another form. If $PA = r$, $PB = r'$, $AB = 2c$,

$$V = \gamma k\rho\log\frac{r+r'+2c}{r+r'-2c},$$

$$\text{or}\quad V = \gamma k\rho\log\frac{a+c}{a-c}, \quad (\beta)$$

where $a =$ semi-axis major of the ellipse described through P with A and B for foci.

The equipotential surfaces are surfaces for which a is constant; they are therefore ellipsoids of revolution having the extremities A and B for foci.

If we assign to V a series of values, the corresponding values of a may be graphically represented. Equation (β) gives

$$\frac{a}{c} = \frac{e^{\frac{V}{2\gamma k\rho}} + e^{-\frac{V}{2\gamma k\rho}}}{e^{\frac{V}{2\gamma k\rho}} - e^{-\frac{V}{2\gamma k\rho}}}.$$

Draw a line Ox and represent a series of values of V by successive lengths measured along it from O. Construct a catenary whose equation is

$$y = \frac{\gamma k\rho}{2}\left(e^{\frac{V}{2\gamma k\rho}} + e^{-\frac{V}{2\gamma k\rho}}\right),$$

O being the origin and Ox the horizontal axis of this catenary.

Along the other axis draw a line parallel to Ox at a distance $c\,(=\frac{1}{2}$ length of bar); then the lengths intercepted on the successive tangents to the catenary between these two parallel lines are the semi-axes of the corresponding ellipses which generate the equipotential surfaces by revolving round AB.

From the value of V given in (β) we can deduce the value of the force-intensity at P. For, the resultant acts in the bisector of the angle APB, and if ds is an element of length of this line at P, $\frac{dV}{ds} = 2\gamma k\rho \frac{c}{a^2-c^2}\frac{da}{ds}$. Now if $\angle APB = 2\phi$, $AP = r$, we have $\frac{dr}{ds} = \cos\phi$. Also $r+r' = 2a$, and since at a point near P on the bisector of APB (tangent to a hyperbola confocal with the ellipse) $r-r'$ is constant, we have $dr = dr'$, therefore $\frac{da}{ds} = \cos\phi$, and

$$\frac{dV}{ds} = \frac{2\gamma k\rho c}{a^2-c^2}\cos\phi.$$

Again, $\cos\phi = \sqrt{\frac{a^2-c^2}{rr'}}$, by elementary trigonometry, and if p is the perpendicular from P on AB, we have $2cp = rr'\sin 2\phi$; therefore

$$\frac{dV}{ds} = \frac{2\gamma k\rho\sin\phi}{p},$$

which is the value already found (Art. 317).

A particular case must now be noted. If the bar is infinitely long, the expression (α) gives $V=\infty$, i.e. the sum $\int\frac{dm}{r}$ is really infinite in any given position of P. On the other hand, we can see that the work which would be done by the attraction of the bar in bringing the condensed unit mass from infinity up to the finite position P is *not* ∞. For if we imagine the bar to be a circle of immense diameter OO', the point O being near us and O' remote, and also that the unit mass is brought from O' up to P, it is quite clear that while P is moving from O' up to the centre of the circle, the attraction of the circle is doing *negative* work, the resultant force being all through this motion directed towards O'; and that when P leaves the centre and moves towards O, the attraction does *positive* work; so that the total amount done in the motion from O' to the final position P is numerically equal to that which would be done in bringing the unit simply from O to P—which obviously is far from being ∞.

But observe that, with some of the attracting mass contemplated as existing at infinity, we are no longer to regard the integral $\int\frac{dm}{r}$, in a given finite position P, extended over the body, as the work done by the attraction in bringing the unit mass from infinity to P.

That, in the case of an infinitely long bar, the amount of work done by the attraction in bringing the unit from a perpendicular distance q to a perpendicular distance p is simply

$$2\gamma k\rho \log_e \frac{q}{p}, \tag{γ}$$

may be seen by taking the resultant force, R, at any distance, x, viz. $\frac{2\gamma k\rho}{x}$, and taking $-\int R dx$.

We must bear in mind that (γ) will not hold for positions of P very close to the surface of the bar, i.e. for very small values of p; because for such points the linear dimensions of the transverse section become comparable with the distances of P from the various points in the section—as has been already pointed out in Art. 317.

3. Without any consideration of force or of work done, show that the difference,

$$\left(\int\frac{dm}{r}\right)_P - \left(\int\frac{dm}{r}\right)_Q,$$

of the summations over an infinite bar with reference to any two finite positions P and Q is finite and, when multiplied by the gravitation constant, equal to the expression (γ).

Instead of finding each integral separately, perform the summation in a different order. Thus, M being any point on the bar, and $OM = s$ (Fig. 276), take at once the difference of effects at P and Q produced by the particle at M. This gives

$$\gamma k\rho\left(\frac{1}{\sqrt{p^2+s^2}} - \frac{1}{\sqrt{q^2+s^2}}\right)ds.$$

Integrating this from $s = -l$ to $s = +l$, we get

$$\gamma k\rho \log_e\left(\frac{\sqrt{p^2+l^2}+l}{\sqrt{q^2+l^2}+l}\cdot\frac{\sqrt{q^2+l^2}-l}{\sqrt{p^2+l^2}-l}\right),$$

which assumes an indeterminate form when $l = \infty$; but a simple binomial development of $\left(1+\frac{q^2}{l^2}\right)^{\frac{1}{2}}$ shows at once the true value to be (γ).

4. To find the potential at any point on the axis of a thin uniform circular plate.

With the notation of Art. 318, the potential at P due to the ring of radius r is $\gamma \cdot \frac{2\pi\rho\tau r dr}{z\sec\phi}$, or $\gamma\frac{2\pi\rho\tau z\sin\phi}{\cos^2\phi}d\phi$; therefore the potential produced by the whole plate is

$$2\pi\gamma\rho\tau z(\sec\alpha - 1),$$

or

$$2\pi\gamma\rho\tau(\sqrt{z^2+a^2}-z),$$

where a = radius of plate.

5. To find V for a uniform spherical shell.

Firstly, at an internal point, P' (Fig. 277, Art. 319). Breaking up

the shell into elements QR, $Q'R'$ which are thin conical frustums, as in Art. 319, if $d\omega$ is the conical angle subtended by either at P', the volume of the frustum is $\rho\tau . P'Q^2 \sec P'QO . d\omega$, or $\frac{2a\rho\tau . P'Q^2}{QQ'} d\omega$. The potential due to this at P' is $2\gamma a\rho\tau \frac{P'Q}{QQ'} d\omega$. Similarly the potential due to the frustum $Q'R'$ is $2\gamma a\rho\tau \frac{P'Q'}{QQ'} d\omega$; and the sum of these $= 2\gamma a\rho\tau\, d\omega$. Hence

$$V = 4\pi\gamma\rho\tau . a; \tag{1}$$

which shows that V is constant wherever P' may be inside—a result for which we are already prepared, since everywhere inside the resultant attraction $= 0$, and this requires that V is constant.

Secondly, for an external point, P. This may be deduced from the value of V at the inverse point, P'. For, the element contributed by the frustum QR is $\gamma \frac{dm}{PQ}$, where dm = mass of frustum. But $PQ = \frac{D}{a} . P'Q$, therefore the element of potential $= \frac{\gamma a}{D} . \frac{dm}{P'Q}$, which bears the constant ratio, $\frac{a}{D}$, to the potential of the element at P'. Hence the potential of the whole shell at P is $\frac{a}{D}$ times the potential at P', or

$$V = \frac{4\pi\gamma\rho\tau a^2}{D},$$

which is the same as if the shell were condensed into a particle at its centre.

The resultant attraction-intensity at $P = -\frac{dV}{dD}$ (measured towards O), which gives the same result as before.

These results can also be easily deduced analytically by breaking up the shell into zones, as has been done (Art. 319) in calculating the force-intensity at P and P'. Thus, the mass of a zone being, as in Art. 319, $2\pi\rho\tau \frac{a}{c} r dr$ where $r = P'Q$ or PQ, the potential produced by the zone is $2\pi\gamma\rho\tau \frac{a}{c} dr$; and for the internal point the limits of r are $a \pm c$, while for the external point they are $c \pm a$.

The value of V can also be deduced from the differential equation (γ), Art. 329. For V depends solely on the distance, r, of the internal point from the centre, and not on θ or ϕ. Hence (γ) reduces to

$$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0,$$

therefore $r^2 \frac{dV}{dr} = C =$ constant. But at the centre the force-intensity, $\frac{dV}{dr}$, vanishes, $\therefore\ C = 0$, $\therefore\ \frac{dV}{dr} = 0$ everywhere inside,

$$\therefore\quad V \text{ at any point} = V \text{ at centre} = \gamma . \frac{\text{mass of shell}}{a}.$$

It follows that for a homogeneous spherical shell contained between a sphere of radius a' and a sphere of radius a, the potential at any point inside the inner sphere (a') is $4\pi\gamma\rho\int_{a'}^{a} r dr$; i.e.

$$V = 2\pi\gamma\rho(a^2 - a'^2),$$

while at an external point at a distance D from the centre

$$V = \tfrac{4}{3}\pi\gamma\rho\frac{a^3 - a'^3}{D}.$$

At a point inside the matter of the shell, at a distance c from the centre

$$V = 2\pi\gamma\rho(a^2 - c^2) + \tfrac{4}{3}\pi\gamma\rho\frac{c^3 - a'^3}{c}.$$

6. To find V at any point for a solid homogeneous sphere. If the point is outside the sphere (radius a),

$$V = \tfrac{4}{3}\pi\gamma\rho\frac{a^3}{D}.$$

If it is inside, at a distance c from the centre, add the potential due to the shell contained between the surface of the given sphere and that of the sphere of radius OP', to the potential due to the solid sphere of radius OP'. Thus

$$V = 2\pi\gamma\rho(a^2 - c^2) + \tfrac{4}{3}\pi\gamma\rho c^2$$
$$= 2\pi\gamma\rho a^2 - \tfrac{2}{3}\pi\gamma\rho c^2.$$

V can also be obtained from the differential equation (γ), Art. 329. Thus, at any internal point this equation gives, since V is a function of r only,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -4\pi\gamma\rho;$$

$$\therefore\ r^2\frac{dV}{dr} = -\tfrac{4}{3}\pi\gamma\rho r^3 + C.$$

Now $\frac{dV}{dr} = 0$ at the centre, $\therefore\ C = 0$, and $V = -\tfrac{2}{3}\pi\gamma\rho r^2 + C'$.

But at the centre V is easily seen to be $2\pi\gamma\rho a^2$, $\therefore$ this $= C'$.

7. To find V for an infinite homogeneous circular cylinder. If P is outside, $\Delta^2 V = 0$. Use cylindrical co-ordinates. Then V is simply a function of ζ (Art. 329), so that

$$\frac{d^2V}{d\zeta^2} + \frac{1}{\zeta}\frac{dV}{d\zeta} = 0. \tag{1}$$

Therefore by integration

$$\frac{dV}{d\zeta} = \frac{C}{\zeta}.$$

To determine C, suppose P to be very distant from the cylinder, so that the latter may be treated as a thin bar. Then $\frac{dV}{d\zeta}$ is the force-intensity at P, which

$$= -\frac{2\gamma k\rho}{\zeta};\ \therefore\ C = -2\gamma k\rho = -2\pi\gamma\rho a^2,$$

if a = radius of cylinder. Hence

$$\frac{dV}{d\zeta} = -\frac{2\pi\gamma\rho a^2}{\zeta}, \tag{2}$$

which shows that the intensity of attraction at any point outside the cylinder varies inversely as the distance from the axis. Integrating,

$$V = -2\pi\gamma\rho a^2 \log_e \zeta + C;$$

and to determine C, let the point P be supposed so far from the cylinder that the latter may be taken as a mere bar, or wire. Now in this case V is given in Example (2), and since A and B are both zero, $V = \infty$, therefore $C = \infty$.

When none of the attracting matter is at infinity, V is, as has been explained, the work done in bringing a condensed unit mass from infinity to the position P, but it ceases to have this meaning when attracting matter is contemplated as existing at infinity. The summation $\int \frac{dm}{r}$ for an infinitely long bar is, in every position of P, really infinite. But if we are concerned only with the amount of work done in bringing the unit mass from one *finite* position, Q, to another, P, we can easily show that the difference

$$\left(\int \frac{dm}{r}\right)_P - \left(\int \frac{dm}{r}\right)_Q$$

is finite, notwithstanding that each integral itself is of infinite magnitude (see Example 3).

Moreover, the supposition itself on which the equation for V is (1) falls to the ground; for it is only for points *finitely* distant from the cylinder that V depends simply on ζ.

Hence instead of choosing infinity as the zero position of P we must choose some other. We may choose a position on the surface of the cylinder, and define the potential at P as the work done by the attraction in conveying a gramme mass from P to the surface of the cylinder. With this definition, we have

$$V = 2\pi\gamma\rho a^2 \int_a^\zeta \frac{d\zeta}{\zeta}$$

$$= 2\pi\gamma\rho a^2 \log_e \frac{\zeta}{a},$$

and now $\frac{dV}{d\zeta}$ will be the force-intensity in the negative sense of ζ.

If P is inside the substance of the cylinder, (1) must, by Art. 329, be replaced by

$$\frac{1}{\zeta}\frac{d}{d\zeta}\left(\zeta\frac{dV}{d\zeta}\right) = -4\pi\gamma\rho;$$

$$\therefore \quad \zeta\frac{dV}{d\zeta} = -2\pi\gamma\rho\zeta^2 + C,$$

and since $\frac{dV}{d\zeta} = 0$ on the axis, $C = 0$, $\therefore \; \frac{dV}{d\zeta} = -2\pi\gamma\rho\zeta$,

$$\therefore \; V = -\pi\gamma\rho\zeta^2 + C' = \pi\gamma\rho(a^2 - \zeta^2).$$

8. To find the potential and attraction-intensity at any point produced by a spherical shell whose thickness (or density) varies inversely as the cube of the distance from a given point.

Let the thickness of the shell at any point Q (Fig. 281) vary inversely as the cube of the distance of Q from the given external point O, and let the potential at any internal point, P, be required.

Produce OP to S so that $OP.OS = OQ.OR =$ square of tangent from O. Imagine the spherical surface broken up into elements by cones described about S, each with very small conical angle, $d\omega$. One of these cones is represented by the line RSU, which must be understood to stand for a slender double cone of rays with vertex S. Joining O to all points on the contours of the small areas at R and U by lines represented by OR and OU, we obtain small elements of area at Q and T; joining the contours of these to S, we obtain two small frustums of cones at Q and T.

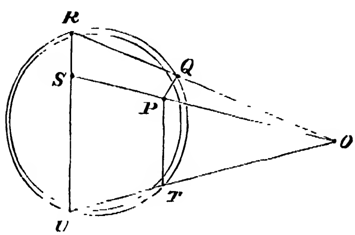

Fig. 281.

It is the sum of the potentials at P produced by these frustums that we shall take; and it is clear that the assemblage of such frustums as those at Q and T will exhaust the whole shell.

Let a be the radius of the sphere. Then (Art. 316) the cone of rays through S cuts off an area $\frac{2a}{RU} \cdot SR^2 . d\omega$ at R. Also, since the elements of area at R and Q are equally inclined to the line OQR, they are to each other in the ratio $OR^2 : OQ^2$; therefore the element of area at Q is

$$\frac{OQ^2}{OR^2} \cdot \frac{2a}{RU} \cdot SR^2 d\omega.$$

But $\frac{SR}{RO} = \frac{QP}{PO}$ (from the similar triangles ROS and POQ); therefore this element of area $= OQ^2 . \frac{PQ^2}{PO^2} \cdot \frac{2a}{RU} d\omega$. Let the thickness at Q be $\frac{k}{OQ^3}$, where k is a constant. Then the potential of this element at P is $2\gamma ak\rho \frac{PQ.d\omega}{RU.OQ.PO^2}$; and similarly the potential at P due to the element at T is $2\gamma ak\rho \frac{PT.d\omega}{RU.OT.PO^2}$; and the sum of these

$$= \frac{2\gamma ak\rho d\omega}{RU.PO^2} \cdot \frac{PQ.OT + PT.OQ}{OQ.OT} = \frac{2\gamma ak\rho d\omega}{RU.PO} \cdot \frac{QT}{OQ.OT}$$

(by Ptolemy's theorem for the quadrilateral $OQPT$, which is inscribable in a circle). But the triangles OQT and OUR are similar,

$$\therefore \frac{QT}{OQ} = \frac{RU}{UO};$$

hence the above sum $= \dfrac{2\gamma a k \rho\, d\omega}{t^2 . OP}$, where $t^2 = OT . OU =$ square of tangent.

Now the whole shell is exhausted by summing $d\omega$ from 0 to 2π, and as the multiplier of $d\omega$ is constant, we have

$$V = \frac{4\pi\gamma a k \rho}{D^2 - a^2} \cdot \frac{1}{OP}, \tag{1}$$

where D is the distance of O from the centre. Hence the remarkable result that the potential at any internal point varies inversely as its distance from O.

We shall call O the *inducing point.*

The mass of the shell is easily found. For (Art. 319) the area of the belt generated by the revolution of the element of length at Q about the line joining O to the centre is $2\pi \dfrac{a}{D} r dr$, where $r = QO$. Hence the mass of this $= \dfrac{2\pi a k \rho}{D} \dfrac{dr}{r^2}$, and if $M =$ mass of shell

$$M = \frac{4\pi k \rho a^2}{D(D^2 - a^2)}, \tag{2}$$

since the limits of r are $D \pm a$.

Hence from (1) and (2)

$$V = \gamma \frac{\frac{D}{a} \cdot M}{OP}, \tag{3}$$

which shows that *the potential at any internal point is the same as if a mass greater than that of the shell in the ratio* $\dfrac{D}{a}$ *were concentrated at the inducing point.*

Of course it follows that the attraction of the shell on a particle at P acts in the line PO, and is equal to

$$\gamma \frac{\frac{D}{a} \cdot M}{OP_2},$$

per unit mass at P.

The inducing point being still external, let the attracted particle be also external to the shell—at P', suppose.

Take the inverse point P, which will be internal. Then since $\dfrac{QP'}{QP}$ is constant, $= \dfrac{R}{a}$, where R is the distance of P' from the centre, it follows that V at $P' = V$ at P multiplied by $\dfrac{a}{R}$,

$$\therefore\ V = \gamma \frac{\frac{D}{R} \cdot M}{OP}.$$

But instead of OP we can put $\dfrac{D}{R} O'P$, where O' is the inverse of O, on account of similar triangles. Hence at any external point

$$V = \gamma \frac{M}{O'P}. \tag{4}$$

so that *for an external point the mass of the shell may be concentrated at the internal point which is the inverse of the inducing point, and the attraction is directed towards this inverse point.*

Finally, consider the case in which the inducing point is inside. This is at once reducible to the case in which the inducing point is outside, by taking the inverse point. Let O be the inducing point, and O' its inverse. Let the attracted particle, P, be inside, and let the thickness at any point, Q, of the shell be $\frac{k}{OQ^3}$; thus it will also be $\frac{kD^3}{a^3} \cdot \frac{1}{O'Q^3}$, where D is the distance of O' from the centre; so that the values of V and M are given by (1) and (2) in which we replace k by $\frac{kD^3}{a^3}$; and we have the result (3), viz.

$$V = \gamma \frac{\frac{D}{a} \cdot M}{O'P}, \tag{5}$$

which shows that the attraction is directed to O'.

Let P be external, while O is internal. Take the inverses of both, so that P' is internal and O' external.

If V' is the potential at P', we have by (3),

$$V' = \gamma \frac{\frac{D}{a} \cdot M}{O'P'}.$$

But if V is the potential at P, we have $\frac{V'}{V} = \frac{R}{a}$, where R is the distance of P from the centre; also, as we are finally concerned with P and not with P', we shall substitute OP for $O'P'$ by the equation $\frac{O'P'}{D} = \frac{OP}{R}$. Hence

$$V = \gamma \frac{M}{OP}. \tag{6}$$

Four different cases may therefore arise, viz. inducing and attracted point both on same side of surface, or on opposite sides; and summarizing the results, we may say that the effect on the attracted particle is always the same as if a certain mass were condensed at a point on the opposite side of the surface; this mass is always *equal* to that of the shell when the attracted particle is outside, and always greater than that of the shell when the particle is inside. The point at which the shell may be condensed is always either the given inducing point or its inverse.

The solution of this question by the ordinary application of the Integral Calculus would be very much more difficult than the simple and elegant solution here given, which is due to Sir William Thomson. (See his *Papers on Electrostatics and Magnetism*, pp. 60, &c.; or Thomson and Tait's *Nat. Phil.*, vol. i, part ii.)

Newton also made use of the relation between inverse points in discussing the attraction of a sphere (see Book I of the *Principia*, Prop. 82).

9. To find the attraction of a thin circular plate of uniform thickness and density on a particle in its plane, the law of attraction being that of the inverse cube of the distance.

Let P (Fig. 282) be the position of the attracted particle, whose mass may be supposed to be one unit.

From P draw two very close radii vectores intercepting a narrow strip of the plate between them.

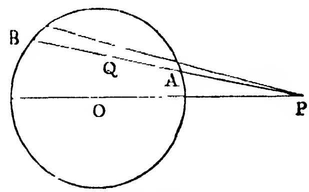

Fig. 282.

Let O be the centre of the plate, let θ be the angle OPA made by one of the radii vectores, and let $\theta+d\theta$ be the angle made by the other, with OP. Let Q be a point on PA, and $PQ=r$. Then the mass of the element at Q included between circles of radii r and $r+dr$ described with P as centre is

$$k\rho r\,dr\,d\theta,$$

k and ρ being the thickness and density of the plate.

The attraction of this element on P resolved along PO is

$$\gamma\frac{k\rho\,dr\,d\theta}{r^2}\cos\theta;$$

hence the resultant attraction is

$$\gamma k\rho\iint\frac{dr\,d\theta}{r^2}\cos\theta,$$

the integrations in r being performed from $r=PA$ to $r=PB$, and those in θ from $\theta=-\sin^{-1}\frac{a}{c}$ to $\theta=\sin^{-1}\frac{a}{c}$, where a is the radius of the plate and $c=OP$, the extreme values of θ corresponding to the two tangents that can be drawn from P to the circle.

Now denoting PA by r_1 and PB by r_2, and integrating first with respect to r, we see that the attraction is

$$\gamma k\rho\int\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\cos\theta\,d\theta.$$

The values of r_1 and r_2 are given by the equation

$$r^2-2cr\cos\theta+c^2-a^2=0,$$

$$\therefore\ \frac{1}{r_1}-\frac{1}{r_2}=\frac{2\sqrt{a^2-c^2\sin^2\theta}}{c^2-a^2}.$$

Hence the attraction is

$$\frac{2k\rho\gamma}{c(c^2-a^2)}\int_{-a}^{a}\sqrt{a^2-t^2}\,dt,\ \text{or}\ \frac{\pi k\rho\gamma a^2}{c(c^2-a^2)},$$

where t is put for $c\sin\theta$.

In this case we might have found the attraction from the potential. The latter is easily found by dividing the plate into rings with O as centre. If r is the radius of one of these rings, we have

$$V = \frac{\gamma k \rho}{2} \iint \frac{r d\theta\, dr}{r^2 - 2cr\cos\theta + c^2}.$$

Integrating first from $\theta = 0$ to $\theta = \pi$, and doubling the result, we have

$$V = \pi k \rho \gamma \int \frac{r dr}{c^2 - r^2},$$

in which r runs from 0 to a. Hence

$$V = \frac{\pi k \rho \gamma}{2} \log \frac{c^2}{c^2 - a^2}.$$

But V may also be easily found from the attraction, thus:

$$\frac{dV}{dc} = -\frac{\pi k \rho \gamma a^2}{c(c^2 - a^2)},$$

$$\therefore\ V = \frac{\pi k \rho \gamma}{2} \log \frac{c^2}{c^2 - a^2} + \text{const.}$$

Now, since $V = \frac{\gamma}{2} \int \frac{dm}{r^2}$, it is clear that at infinity $V = 0$, or $V = 0$ when $c = \infty$. This gives the const. $= 0$,

$$\therefore\ V = \frac{\pi k \rho \gamma}{2} \log \frac{c^2}{c^2 - a^2},$$

10. If V_n and V_{n-2} denote the potentials of an attracting mass when the law of attraction is the n^{th} and $(n-2)^{\text{th}}$ power of the distance, respectively, prove that

$$V_{n-2} = \frac{\nabla^2 V_n}{(n-1)(n+2)},$$

where $\nabla^2 \equiv \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$, the co-ordinates of the attracted particle being x, y, z.

We have

$$V_n = -\frac{\lambda}{n+1} \int r^{n+1} dm,$$

where λ is a constant. Therefore

$$\frac{dV_n}{dx} = -\lambda \int (x - x') r^{n-1}\, dm,$$

and

$$\frac{d^2 V_n}{dx^2} = -\lambda \int \{r^{n-1} + (n-1)(x - x')^2 r^{n-3}\}\, dm.$$

Adding to this the similar values of $\frac{d^2 V}{dy^2}$ and $\frac{d^2 V}{dz^2}$, we have

$$\nabla^2 V_n = (n-1)(n+2)\, V_{n-2}.$$

This equation enables us, generally, to find the potential for the $(n-2)^{\text{th}}$ power of the distance when that for the n^{th} is known; but it fails in two most important cases, namely, when $n = 1$ and when $n = -2$.

If the attracting mass is a plate, $r^2 = (x-x')^2+(y-y')^2$, and the result is easily proved to be

$$\nabla^2 V_n = (n^2-1)V_{n-2}.$$

In the last example we find the Potential of a circular plate for the inverse third power; hence we have at once the Potentials, and therefore the attractions for the inverse fifth, seventh, &c., powers of the distance.

11. Calculate the attraction of a uniform spherical shell of small thickness on an external particle when the attraction varies as the n^{th} power of the distance.

Using the expression (A), Art. 320, for the element of surface, and assuming the law of attraction to be λr^n, we have

$$\begin{aligned} V &= -\frac{2\pi\lambda\rho\tau}{n+1}\frac{a}{D}\int_{D-a}^{D+a} r^{n+2}\,dr \\ &= -\frac{2\pi\lambda\rho\tau a}{(n+1)(n+3)D}\left[(D+a)^{n+3}-(D-a)^{n+3}\right], \end{aligned}$$

where D is the distance of the point from the centre.

If we wish to find the attraction of a full sphere of radius r, we observe that τ is da, and we integrate this expression from $a = 0$ to $a = r$.

In each case the attraction towards the centre is $-\dfrac{dV}{dD}$.

12. From the theorem of Gauss (Art. 331) deduce the following result—*the mean Potential over a spherical surface due to matter entirely outside the sphere is equal to the Potential of this matter at the centre of the sphere.* (Gauss, Papers on Forces varying inversely as the square of the distance, Taylor's *Scientific Memoirs*, vol. iii, part x.)

For, let mass of uniform density, ρ, and small uniform thickness, τ, be supposed to be distributed on the sphere; let dS be an element of its surface at any point P, V the Potential at P due to the external attracting mass, and a the radius of the sphere. Then, since the Potential of a shell at an external point whose distance from the centre is r

$$= \frac{4\pi\gamma\rho\tau a^2}{r},$$

it follows that if dm is an element of the attracting matter,

$$\rho\tau\int V\,dS = 4\pi\gamma\rho\tau a^2\int\frac{dm}{r} = 4\pi\rho\tau a^2 V_0,$$

if V_0 is the Potential at the centre of the sphere. Hence

$$\frac{\int V\,dS}{4\pi a^2} = V_0,$$

which proves the proposition, since $\int V\,dS$ divided by the whole surface of the sphere is the mean value of the Potential over its surface.

More elementary proof. Let there be a particle of mass m outside

a spherical surface of radius a at a distance D from its centre. The mean value of the Potential over the sphere is $\frac{\gamma m}{4\pi a^2}\int\frac{dS}{r}$, where r is the distance from m of the element dS of surface. But (Art. 319) $dS = 2\pi\frac{a}{D}rdr$, and the limits of r are $D \pm a$. Hence this mean value is

$$\frac{\gamma m}{D},$$

i. e. the Potential at the centre; and the result therefore holds for any assemblage of external particles.

13. Find an approximate value of the Potential of any solid mass at a very distant point.

Let G be the centre of mass of the solid body, P the distant point, P' any point in the mass at which the element of mass is dm. Take G as origin and GP as axis of x; let $GP = r$, $GP' = r'$, and let the x of P' be x'.

$$\text{Then } \frac{1}{\gamma}V = \int\frac{dm}{\sqrt{r^2-2rx'+r'^2}} = \frac{1}{r}\int\left(1-2\frac{x'}{r}+\frac{r'^2}{r^2}\right)^{-\frac{1}{2}}dm$$

$$= \frac{1}{r}\int\left(1+\frac{x'}{r}-\frac{r'^2}{2r^2}+\frac{3}{2}\cdot\frac{x'^2}{r^2}\right)dm,$$

neglecting all higher powers of $\frac{r'}{r}$ than the second.

Now $\int x'dm = 0$, and if we denote by λ and λ' the radii of gyration of the solid about the axes of y and z, and by k its radius of gyration about GP, we have

$$\int r'^2dm = M\frac{\lambda^2+\lambda'^2+k^2}{2},\quad \int x'^2dm = M\frac{\lambda^2+\lambda'^2-k^2}{2},$$

where $M =$ mass of body.

$$\text{Hence}\qquad V = \frac{M}{r}\left(1+\frac{\lambda^2+\lambda'^2-2k^2}{2r^2}\right).$$

But if k_1, k_2, k_3 are the principal radii of gyration at G, we have $\lambda^2+\lambda'^2+k^2 = k_1^2+k_2^2+k_3^2$; therefore

$$V = \frac{\gamma M}{r}\left(1+\frac{k_1^2+k_2^2+k_3^2-3k^2}{2r^2}\right).$$

By differentiating this with respect to x, y, and z separately, we find the components of attraction in the directions of the principal axes at G on a unit mass at P. For very distant points $V = \frac{\gamma M}{r}$ to a high degree of accuracy.

14. If $V \equiv f(x, y, z)$ be a function satisfying Laplace's equation, $\nabla^2 V = 0$, show that the function $\frac{a}{r}f\left(\frac{a^2x}{r^2}, \frac{a^2y}{r^2}, \frac{a^2z}{r^2}\right)$ will also satisfy it (where $r^2 = x^2+y^2+z^2$).

If O is the origin, P the point (x, y, z), Q a point on OP produced such that $OQ = \frac{a^2}{OP}$, the co-ordinates of Q are $\frac{a^2 x}{r^2}, \frac{a^2 y}{r^2}, \frac{a^2 z}{r^2}$. Let $OQ = \rho$, let (ξ, η, ζ) be the co-ordinates of Q, and let

$$U = \frac{a}{r} f\left(\frac{a^2 x}{r^2}, \frac{a^2 y}{r^2}, \frac{a^2 z}{r^2}\right) = \frac{\rho}{a} f(\xi, \eta, \zeta).$$

Then $\frac{aU}{\rho}$ satisfies the equation

$$\sin\theta \frac{d}{d\rho}\left(\rho^2 \frac{d\frac{U}{\rho}}{d\rho}\right) + \frac{d}{d\theta}\left(\sin\theta \frac{d\frac{U}{\rho}}{d\theta}\right) + \frac{1}{\sin\theta} \frac{d^2\frac{U}{\rho}}{d\phi^2} = 0.$$

But $\rho^2 \frac{d}{d\rho} = -a^2 \frac{d}{dr}$; therefore this equation becomes

$$r \sin\theta \frac{d^2 (Ur)}{dr} + \frac{d}{d\theta}\left(\sin\theta \frac{dU}{d\theta}\right) + \frac{1}{\sin\theta} \frac{d^2 U}{d\phi^2} = 0.$$

The first term being the same as $\sin\theta \frac{d}{dr}\left(r^2 \frac{dU}{dr}\right)$, this equation is, by Art. 329, the equivalent of

$$\frac{d^2 U}{dx^2} + \frac{d^2 U}{dy^2} + \frac{d^2 U}{dz^2} = 0.$$

15. A homogeneous fluid, self-attracting according to the law of nature, completely fills the space between two spherical non-concentric surfaces one of which entirely surrounds the other; find the resultant attraction at any point of the fluid, and also the level surfaces.

Let O be the centre of the larger and O' the centre of the smaller sphere; P any point in the fluid; $OO' = c$; radius of smaller sphere $= b$; $OP = r$, $O'P = r'$; $\rho =$ density of fluid.

To calculate the resultant force at P, imagine that the place of the smaller sphere is occupied with fluid; then the larger is completely full, and there is a force $\frac{4}{3}\pi\gamma\rho r$ in the line PO towards O. Now let the effect of the fluid which we have introduced be annulled by combining with the above force the force exercised at P by a repulsive fluid of same density filling the smaller sphere. This latter force would be $\frac{4\pi\gamma\rho b^3}{r'^2}$; and this would act in the line $O'P$ from O'.

The resultant of these forces is the resultant force at P. If V is the Potential at P,

$$\frac{1}{\gamma} dV = -\frac{4}{3}\pi\rho r dr + \frac{4\pi\rho b^3}{3r'^2} dr';$$

$$\therefore \quad \frac{V}{\gamma} = -\frac{2}{3}\pi\rho r^2 - \frac{4\pi\rho b^3}{3r'} + \text{const.}$$

This value is otherwise evident, since the Potential at a point due

to any attracting bodies is the sum of their separate Potentials at the point. If a is the radius of the larger sphere (see Art. 329),

$$\frac{V}{\gamma} = -\frac{2}{3}\pi\rho r^2 - \frac{4\pi\rho b^3}{3r'} + 2\pi\rho a^2.$$

The level surfaces are given by the equation

$$r^2 + \frac{2b^3}{r'} = \text{const.}$$

16. If two different masses have the same external level surfaces, the values of their Potentials on any one common surface of level are directly proportional to the quantities of the two masses.

Let M and M' be the two masses; let V be the Potential of the first and V' that of the second at any point P outside both. Then

$$\nabla^2 V = 0, \quad \nabla^2 V' = 0. \tag{1}$$

Now since when V is constant, V' is also constant, V' must be some function of V. Let $V' = \phi(V)$. Performing the operation ∇^2 on both sides of this equation, we have

$$\nabla^2 V' = \phi'(V).\nabla^2 V + \phi''(V)\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\}, \tag{2}$$

which (1) reduces to $\phi''(V) = 0$, since the coefficient of $\phi''(V)$ cannot vanish.

Hence $\phi(V) = cV + c'$, $\therefore\ V' = cV + c'$; but since at infinity $V = V' = 0$ (if none of the attracting matter is at infinity),

$$c' = 0; \quad \therefore \quad V' = cV.$$

Again, for very distant points (Example 13),

$$V = \gamma\frac{M}{r} \text{ and } V' = \gamma\frac{M'}{r}.$$

Hence, finally, $$\frac{V'}{M'} = \frac{V}{M}.$$

17. If X_n and X_{n-2} denote the component attractions of a given solid at a given point along a given line when the law of attraction is that of the n^{th} power, and that of the $(n-2)^{\text{th}}$ power, of the distance, respectively, prove that

$$X_{n-2} = \frac{\nabla^2 X_n}{(n-1)(n+2)}.$$

18. Find the attraction of a circular plate of uniform thickness and density on an external particle of unit mass in its plane, the law of attraction being that of the inverse distance.

Ans. The mass of the plate divided by the distance of the particle from its centre, multiplied by a constant.

19. Prove that if a material lamina attract according to the law of the inverse distance and if N is its attraction on a unit mass at any

point of a closed curve, measured outwards along the normal, we shall have

$$\int N ds = 0, \text{ or } = -2\pi\gamma m_i,$$

according as there is no mass or mass m_i inside the closed curve, and hence that $\nabla^2 V = 0$ or $= -2\pi\gamma\rho$.

20. Prove that the values of $\nabla^2 V$ calculated for external points and for internal points do not agree for points on the surface of a solid sphere.

21. Prove that neither Laplace's nor Poisson's equation holds for points on the bounding surface of an attracting solid.

22. If a number of uniform bars of the same section and density form any closed polygon with no re-entrant angle, prove that they produce the same Potential (for the law of the inverse square) at any point inside the polygon as a polygon of bars formed by joining the feet of the perpendiculars from the given point on the sides of the given polygon.

Extend this proposition to any curve.

(See equation (α), Art. 332, Example 2.)

23. If a self-attracting sphere of uniform density and radius a changes to one of uniform density and radius a', find the amount of work done by its mutual attractive forces.

Ans. $$\frac{3}{5}\gamma M^2\left(\frac{1}{a}-\frac{1}{a'}\right),$$

where M is the mass of the sphere, and γ, as usual, the gravitation constant.

24. Two equal uniform bars of given sections and densities are placed parallel to each other and at right angles to the lines joining their extremities; find the amount of work done against their mutual attraction in drawing them a given distance asunder.

Ans. If y is the distance between the bars in any position, l the length of each, m and m' are their masses, the work done in changing the distance from y_1 to y_2 will be the difference of the values of the expression

$$\frac{2\gamma mm}{l^2}\left(y - \sqrt{l^2+y^2} - l\log\frac{\sqrt{l^2+y^2}-l}{y}\right),$$

when y_1 and y_2 are successively put for y.

25. The gravitation Potential of an attracting mass cannot have a maximum or minimum value in empty space.

[Let it have a maximum value at A. Then round A, and indefinitely near it, can be described a closed surface, at every point of which V is less than it is at A. Therefore if dn is an elementary length along the normal (measured outwards) to this surface, $\frac{dV}{dn}$ is negative all over the surface; but $N = \frac{dV}{dn}$; hence equation (2), Art. 234 is contradicted.]

333.] Earnshaw's Theorem. *If a particle is in equilibrium under the action of forces varying according to the law of inverse square of distance, its equilibrium is unstable.*

If its equilibrium were stable for all displacements, positive work would have to be done against the attractive forces, i. e. these forces would for every small displacement do negative work, or, in other words, V must decrease in all directions from the point. V is therefore a maximum at the point—which, by last example, it cannot be. Therefore, &c.

334.] Method of Inversion. Supposing that for any distribution of mass forming either a continuous solid body M (Fig. 283), or a thin shell of any shape, or a series of isolated particles, we know the value of the Potential at any point, A, we may by the aid of an analytical transformation deduce another analogous mass, M', whose Potential at any point, A', may be deduced from the previous Potential.

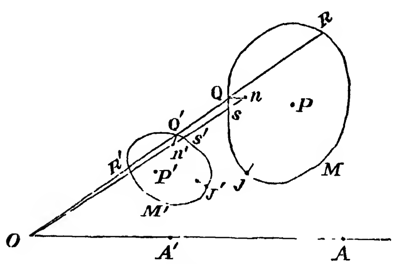

Fig. 283.

Thus, suppose the mass M' to be deduced from the mass M in the following way.

Take any fixed point, O; join it to P, and take the inverse point P', so that

$$rr' = k^2, \tag{α}$$

where $r = OP$, $r' = OP'$. Round P let any very small closed surface be described, and take round P' all the points corresponding to those on this surface. We shall thus get a very small closed surface at P'. Denote the volumes of these elements by $d\Omega$ and $d\Omega'$, respectively. Now $d\Omega$ is filled with a quantity dm of matter belonging to the mass M, and it remains with us to fill $d\Omega'$ with matter according to any law we please. Fill $d\Omega'$ with a quantity dm' bearing to dm the relation

$$\frac{dm'}{dm} = \frac{k}{r}, \text{ or } = \frac{r'}{k}, \tag{β}$$

and let this be done for all the elements, $d\Omega'$, of volume in the derived body M', related as above to the corresponding elements

$d\Omega$ of M. Given the volume-density, ρ, at each point, P, of M; we must now find the volume-density, ρ', at each derived point P'.

Now $dm = \rho d\Omega = \rho r^2 \sin\theta\, dr\, d\theta\, d\phi = -\rho r^2 dr\, d\mu\, d\phi$, in the usual notation of polar formulæ. Similarly, dr' being taken positively without reference to dr,

$$dm' = -\rho' r'^2 dr'\, d\mu\, d\phi;$$

and from (α), if dr and dr' are taken connectedly, we have $r\,dr' + r'\,dr = 0$: hence (β) gives

$$\rho' = \rho \frac{k^5}{r'^5}, \qquad (\gamma)$$

so that if ρ is constant, ρ' will vary inversely as the fifth power of the distance, OP', from O.

Let A be any point, at which the Potential of M is V, and take the inverse point A'. It is required to find V', the Potential of M' at A'.

If dV is the Potential at A produced by the element dm at P,

$$dV = \gamma \frac{dm}{AP},$$

where γ is the gravitation constant. Also if dV' is the Potential at A' due to dm' at P',

$$dV' = \gamma \frac{dm'}{A'P'}.$$

Hence $\dfrac{dV'}{dV} = \dfrac{k}{r} \cdot \dfrac{AP}{A'P'}$. But the triangles PAO and $A'P'O$ are similar, $\therefore$ if $OA = D$, $\dfrac{AP}{A'P'} = \dfrac{D}{r'}$; hence $\dfrac{dV'}{dV} = \dfrac{D}{k}$; and this constant relation holds between the Potentials of all corresponding elements, and therefore between the whole Potentials, so that

$$V' = \frac{D}{k} \cdot V. \qquad (\delta)$$

In this transformation the angle at which any two curves in the original system M intersect is equal to that at which the two derived curves intersect. For, let Q and n be any two very close points in the system M, and let Q' and n' be their inverses. Then the quadrilateral $Qnn'Q'$ is inscribable in a circle, so that Qn and $Q'n'$ are ultimately tangents to the circle and therefore equally inclined to OQ. Similarly, if s and s' are two corresponding points very close to Q and Q', the arcs Qs and $Q's'$ are equally inclined to OQ; therefore the angle between the arcs

Qn and Qs is equal to that between $Q'n'$ and $Q's'$. Hence if the contour M is the outer surface of a shell whose thickness varies in any manner, being Qn at the point Q, the inverse points will trace out the contour M' of another shell, and if n' is the inverse of n, $Q'n'$ will be normal to the new shell, and will, of course, be its thickness at Q'.

By similar triangles $\frac{Qn}{Q'n'} = \frac{OQ}{On'}$, or $= \frac{OQ}{OQ'}$, since Q' and n' are nearly coincident. Hence if τ and τ' are the thicknesses of the shells at corresponding points,

$$\frac{\tau'}{\tau} = \frac{r'}{r}; \tag{ϵ}$$

and hence by (γ)

$$\rho'\tau' = \rho\tau \cdot \frac{k^3}{r'^3}, \tag{ζ}$$

so that if $\rho\tau$ is constant all over the shell M, $\rho'\tau'$ will vary inversely as the cube of the distance from O at every point on the derived shell.

If the mass M forms a spherical shell of uniform thickness and density, its Potential at A is at once known. Hence is known also the potential at A' (any point) due to a spherical shell in which the product $\rho'\tau'$ (which is the mass per unit area of its surface) varies inversely as the cube of the distance from a fixed point, O—a case which has been already discussed (Art. 332, Ex. 8). Supposing M to be a spherical surface whose centre is P, the inverse point, P', is not the centre of the sphere M', but is the inverse of O with respect to the sphere M'. For if J and J' are any corresponding points on the contours, $\frac{JP}{PO} = \frac{P'J'}{J'O}$, and since M is a sphere with centre P, the left-hand side is constant, therefore the right side is constant, and the two points O and P' are well known to be inverse with respect to the spherical locus of J'.

Again, (δ) gives, since $M' = k\int\frac{dm}{r} = k\frac{M}{OP}$,

$$V' = \gamma\frac{M'}{P'A'}, \tag{η}$$

which shows that the level surfaces of M' are spheres round P' as centre; and the result (η) holds both for the case in which M is a uniform spherical shell, and therefore M' a spherical shell

in which the surface-density, $\rho'\tau'$, varies inversely as the cube of the distance from O, and for the case in which M is a solid uniform sphere, and therefore M' a solid sphere in which the density varies inversely as the fifth power of the distance from O.

In this method of transformation we may notice that *the mass of the derived distribution, M', is proportional to the Potential of the given mass at the origin of inversion.* (It is equal to this Potential multiplied by $\frac{k}{\gamma}$.)

335.] **Continuity of the Potential.** The gravitation Potential of any attracting solid mass varies in a continuous manner from point to point in space, whether the points chosen be inside any portion of the mass or outside it.

For, if r be the distance of any element of mass, dm, of the attracting body from P, the point at which the Potential is required, $V = \gamma \int \frac{dm}{r}$. Let P be taken as origin, and let the position of the element dm be defined by the radius vector, r, and two angles, θ and ϕ, and let ρ be the density of the element. Then $dm = \rho r^2 \sin\theta \, dr \, d\theta \, d\phi$, and

$$V = \gamma \int\int\int \rho r \sin\theta \, dr \, d\theta \, d\phi.$$

This form of V shows that even if r is zero, i. e. if P is inside the mass, the value of the Potential is finite, no infinite term being introduced by the infinitely close proximity of P to some of the (infinitely small) elements of mass.

Hence the Potential varies continuously throughout space, and diminishes from the vicinity of the attracting mass towards the space very remote from it in all directions.

The field of attraction of any matter, according to the Newtonian law, may therefore be compared with a country consisting of hills and valleys which vary *gradually*, even though they may rise or fall rapidly in certain places,—precipices and chasms being wholly absent; and in the field of attraction the Potential at each point is the *gravitation level* of the point, and is the analogue of the height above the sea (or other arbitrary) level in the other.

336.] **Continuity of the First Differential Coefficients of Potential.** In a field of attraction *in which every attracting element is one of finite volume-density*, there is likewise a complete continuity of the first differential coefficients of V from points

within to points without the attracting masses. For these first differential coefficients, $\frac{dV}{dx}$, $\frac{dV}{dy}$, $\frac{dV}{dz}$, are simply the components of force-intensity; and if in (3), (4), (5) of Art. 325, we put $\phi(r) = \frac{\gamma}{r^2}$, the elements under the sign of integration never at any point contain r in the denominator, and are therefore never infinite, even when $r = 0$, i.e. when P is inside the mass. Evidently the case would be different for a law of attraction according to a power higher than that of the inverse square.

And the case is different again, even for the Newtonian law, when the attracting matter forms an *infinitely* thin shell with (necessarily) infinitely great volume-density. In this case the force components in *some* directions vary abruptly for a small change of position of the attracted particle P, although in other directions they vary continuously. Of this more hereafter; but the fact is already sufficiently clear in the case (Art. 322) of the normal component of a thin shell.

337.] **Discontinuity of its Second Differential Coefficients.**

Since $V = \gamma \int \frac{dm}{r}$, we have $\frac{d^2V}{dx^2} = \gamma \int \frac{d^2\left(\frac{1}{r}\right)}{dx^2} dm$, the co-ordinates of the point, P, at which the Potential is V, being x, y, z.

Now if (x', y', z') are the co-ordinates of dm, and (x, y, z) those of P, we have

$$\frac{d^2 r}{dx^2} = \frac{1}{r} - \frac{(x-x')^2}{r^3};$$

and since

$$\frac{1}{\gamma}\frac{d^2V}{dx^2} = \int \left\{ \frac{2}{r^3}\left(\frac{dr}{dx}\right)^2 - \frac{1}{r^2}\frac{d^2 r}{dx^2} \right\} dm,$$

$$\therefore \quad \frac{1}{\gamma}\frac{d^2V}{dx^2} = \int \left\{ \frac{3(x-x')^2}{r^5} - \frac{1}{r^3} \right\} dm. \qquad (1)$$

Similarly

$$\frac{1}{\gamma}\frac{d^2V}{dy^2} = \int \left\{ \frac{3(y-y')^2}{r^5} - \frac{1}{r^3} \right\} dm, \qquad (2)$$

$$\frac{1}{\gamma}\frac{d^2V}{dz^2} = \int \left\{ \frac{3(z-z')^2}{r^5} - \frac{1}{r^3} \right\} dm. \qquad (3)$$

If in these expressions we substitute for $x-x'$, $y-y'$, $z-z'$, and dm, as in Article 335, we have

$$\frac{1}{\gamma}\frac{d^2V}{dx^2} = \int (3\sin^2\theta\cos^2\phi - 1)\frac{\rho}{r}\sin\theta \, dr \, d\theta \, d\phi;$$

hence, when $r = 0$, i.e. when P is inside the attracting mass, the expression under the integral sign becomes infinite, and the value of $\frac{d^2 V}{dx^2}$ ceases to be continuous from points inside to points outside the mass.

Fig. 284 represents the values of V, $\frac{dV}{dx}$, and $\frac{d^2 V}{dx^2}$, when the attracting solid is that contained between two concentric spherical surfaces whose radii are Oa' and Oa, and the point P occupies positions along a fixed diameter, Ox, varying from O to infinity. The distance of P from O is here denoted by x, and the values of V are given by the ordinates (distances from Ox) of the continuous curve $ABCD$, of which the portion AB is a right line corresponding to the constant potential within the inner surface.

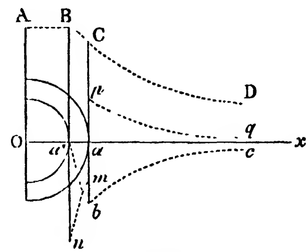

Fig. 284.

The values of $\frac{dV}{dx}$ are given by the ordinates of the continuous curve $Oa'bc$, of which Oa' corresponds to the constant zero value within the inner surface.

The values of $\frac{d^2 V}{dx^2}$ are given by the ordinates of the discontinuous curve $Oa'nmpq$.

From Ex. 5, Art. 332, when P is completely outside the mass, we have $\frac{d^2 V}{dD^2} = \frac{8\pi\gamma\rho\,(a^3 - a'^3)}{3\,D^3}$, and when P is inside the shell between the two surfaces, $\frac{d^2 V}{dD^2} = -\frac{4\pi\gamma\rho}{3}\left(1 + \frac{2a'^3}{D^3}\right)$.

By putting $D = a$ in the first of these values we have the value, ap, of $\frac{d^2 V}{dD^2}$ when P comes to the outer surface from the outside; and putting $D = a$ in the second we have the (negative) value, am, of $\frac{d^2 V}{dD^2}$ when P comes to this surface from the inside.

The above figure is copied from Thomson and Tait's *Nat. Phil.*

338.] Lines and Tubes of Force. If at any point, P, in the field of attraction an elementary length is drawn in the direction of the resultant attraction at P, and if this is prolonged at each point P', P'',... so as to be in the direction of the resultant attraction at all points, P', P'',... along it, we obtain a continuous curve which is called a *line of force*. A line of force,

then, is a curve such that its tangent at every point coincides with the direction of resultant attraction at that point.

If the field is mapped out by a series of equipotential surfaces (Art. 328), every line of force will cut every equipotential surface which it meets at right angles, since (Art. 328) at every point on such a surface the resultant force acts in the normal to the surface.

Let P be any point in the field; at P describe any very small closed curve whatsoever; through each point on this curve draw the line of force and prolong it indefinitely. We thus get what is called a *tube of force*.

These terms are due to Faraday.

339.] **Surface-integral for a Tube of Force.** Let $PAQB$ represent any portion of a tube of force, P and Q being elements of two level surfaces intercepted by the tube. Then the attraction on a unit mass at P is normal to the section P, and the attraction on a unit mass at Q is normal to the section Q, while at every point, A or B, on every portion of the lateral surface of the tube the attraction is wholly tangential to the surface.

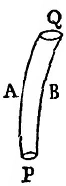

Fig. 285.

Let F be the force at P, F' that at Q, and ω and ω' the areas of the sections P and Q. Then, supposing that the tube contains none of the attracting matter, equation (2) of Art. 324 gives

$$F\omega - F'\omega' = 0, \tag{1}$$

since the only portions of the closed surface $PAQB$ which contribute elements to the surface-integral of normal attraction are the sections P and Q.

Hence, *at all points in empty space on a given line of force the resultant attraction-intensities are inversely proportional to the normal sections of the same tube of force at these points.*

This simple theorem gives the law of attraction very readily in certain cases. For example, let the attracting body be a sphere whose density is the same at the same distance from its centre. Then the lines of force are obviously right lines drawn from its centre; the tubes are therefore cones whose vertices are the centre, and since the normal sections of these cones are directly as the squares of their distances from the centre, the attraction of the sphere at any external point is inversely proportional to the square of its distance from the centre.

Again, let the attracting body be an infinite cylinder whose density is the same at the same distance from its axis. Here the lines of force are right lines emanating from the axis perpendicularly, the tubes become wedges, and the areas of their normal sections are directly proportional to their distances from the axis; hence the attraction of an infinite cylinder at an external point is inversely proportional to its distance from the axis.

Finally, for an infinite attracting plate, the tubes are cylinders and the attraction is constant at all points in empty space.

If the tube of force contain within it a quantity of the attracting matter whose mass is dq, we have by (2) of Art. 324

$$F\omega - F'\omega' = 4\pi\gamma\, dq. \tag{2}$$

This equation can in like manner be employed to find the resultant force inside a sphere, a cylinder, or a plate.

In the case of a sphere of uniform density, let the tube be contained between the spheres of radii r and $r+dr$. Then $dq = \rho\,\omega\, dr$, ρ being the density at the attracted point, and (2) becomes

$$d(F\omega) = 4\pi\gamma\rho\,\omega\, dr,$$

or

$$d(Fr^2) = 4\pi\gamma\rho\, r^2 dr,$$

since ω is proportional to r^2. Integrating this last equation,

$$Fr^2 = \frac{3}{4}\pi\gamma\rho\, r^3 + C.$$

Now F is evidently zero at the centre, therefore $C = 0$, and

$$F = \frac{4}{3}\pi\gamma\rho\, r.$$

For a point inside an infinite cylinder at a distance r from the axis we have, since ω is ultimately a rectangle of breadth proportional to r,

$$d\,(Fr) = 4\pi\gamma\rho\, r\, dr,$$

$$\therefore\quad F = 2\pi\gamma\rho\, r.$$

In general, if the tube is terminated by two level surfaces whose distance measured along the lines of force forming the tube is ds, we have $dq = \rho\,\omega\, ds$, and (2) gives for the determination of F

$$d\,(F\omega) = 4\pi\gamma\rho\,\omega\, ds.$$

340.] **Unit Tube of Force.** If at any point P we draw a tube of force such that the product of the force-intensity and the area of the normal section is unity, the tube is called a *unit tube*. Thus, in C. G. S. measures, if the product of the force-intensity,

expressed in dynes per gramme mass, and the area of the normal section, expressed in square centimetres, is numerically unity, the tube is a unit tube.

Section III.

The Attraction of Ellipsoids. [*Method of Chasles.*]

341.] **Shell bounded by Similar Surfaces.** Let $vr'p'$ and rqp be two concentric, similar, and similarly situated surfaces whose normal distance from each other is at all points very small. Suppose the space between these surfaces to be filled by attracting matter of uniform density, and let O be an attracted particle in the interior of the shell. With O as vertex let any slender cone be described, intercepting on the shell two frustums whose thicknesses measured along the generator pr of the cone are pp' and rr'. Then, since by the property of similar, similarly situated, and concentric surfaces of the second degree, the intercepts pp' and rr' are equal whatever be the direction of the line pr, we see by Art. 318 that the attractions of these frustums on O are equal and opposite. Hence the corresponding frustums of all such cones exert equal and opposite attractions on O; and the resultant attraction of the shell on any internal particle is therefore zero.

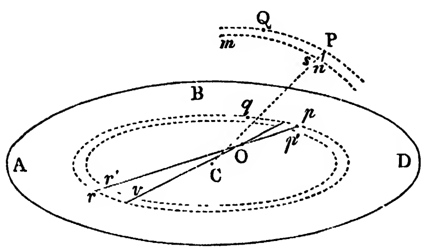

Fig. 286.

Hence, if the law of attraction is that of nature, *every shell of uniform density and small thickness, bounded by similar, similarly situated, and concentric ellipsoidal surfaces produces a constant Potential at all points in its interior, and exerts, therefore, at these points no attraction.*

The same is true for a solid of uniform density and any thickness bounded by two similar, similarly situated, and concentric ellipsoidal surfaces, since the thicknesses of the frustums intercepted between its bounding surfaces will still be equal.

342.] Corresponding Points on Confocal Ellipsoids. Let rpq and PQ (Fig. 286) be two confocal ellipsoids, let the axes of the first be α', β', γ', and those of the second α, β, γ, let the co-ordinates of a point p on the first be x', y', z', and those of a point P on the second x, y, z. Then, if

$$\frac{x}{\alpha}=\frac{x'}{\alpha'},\quad \frac{y}{\beta}=\frac{y'}{\beta'},\quad \frac{z}{\gamma}=\frac{z'}{\gamma'},$$

the points P and p are called *corresponding* points on the ellipsoids. Also, let Q and q be two other corresponding points. Then it is easy to prove that the distance Pq is equal to the distance Qp. (Salmon's *Geometry of Three Dimensions.*)

343.] External Potential of an Ellipsoidal Shell. Let it be required to find the Potential at an external point, P, of a shell bounded by the similar, similarly situated, and concentric ellipsoids $vr'p'$ and rqp. Through the point P describe an ellipsoid, PQ, confocal with rqp, and describe also an ellipsoid, msn, confocal with $vr'p'$ and similar to PQ. This latter surface is completely determinate, since its axes must be $\mu\alpha, \mu\beta, \mu\gamma$, and since $\mu^2(\alpha^2-\beta^2)$ must be equal to $\mu'^2(\alpha'^2-\beta'^2)$, where $\mu'\alpha', \mu'\beta'$, $\mu'\gamma'$ are the (given) axes of the ellipsoid $vr'p'$; or $\mu=\mu'$, since $\alpha^2-\beta^2=\alpha'^2-\beta'^2$.

Now at q draw the normal distance, dn', which separates the surfaces rqp and $vr'p'$, and about q describe on the ellipsoid rqp any small closed curve whose area is dS'. Round Q, on the surface QP, describe the small closed curve, of area dS, which consists of points corresponding to those forming dS'; and let dn be the normal distance between the surfaces QP and msn. We shall now prove that the elements of volume $dn\,.\,dS$ and $dn'.\,dS'$, which we may denote by $d\omega$ and $d\omega'$, respectively, are connected by the equation

$$\frac{d\omega}{\alpha\beta\gamma}=\frac{d\omega'}{\alpha'\beta'\gamma'}. \tag{1}$$

Let x', y', z' be the co-ordinates of q with reference to the principal axes of the ellipsoids, and let $dx'\,dy'$ be the projection of dS' on the plane xy. Then, since the cosine of the angle between the normal at q and the axis of z is $\frac{p'z'}{\gamma'^2}$, where p' is the perpendicular from C on the tangent plane at q, we have

$$dS'=\frac{\gamma'^2}{p'z'}\,dx'\,dy'.$$

Now since the surfaces rqp and $r'r'p'$ are similar, we have

$$\frac{dn'}{p'} = 1-\mu,$$

$$\therefore \quad dn'.dS' = (1-\mu)\frac{\gamma'^2}{z'}dx'dy'. \tag{2}$$

Similarly
$$dn.dS = (1-\mu)\frac{\gamma^2}{z}dxdy, \tag{3}$$

where x, y, z are the co-ordinates of Q. But since

$$\frac{dx}{\alpha} = \frac{dx'}{\alpha'}, \quad \frac{dy}{\beta} = \frac{dy'}{\beta'},$$

these equations give (1) at once by division. Moreover the Potential at P due to the element of mass $\rho d\omega'$ at q is proportional to $\frac{\rho d\omega'}{Pq}$, while the Potential at p due to the element $\rho d\omega$ at Q is proportional to $\frac{\rho d\omega}{Qp}$; and since $Pq = Qp$,

$$\frac{\text{Potential at } P \text{ due to element of mass at } q}{\text{Potential at } p \text{ due to corresponding element at } Q} = \frac{d\omega'}{d\omega}$$

$$= \frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma} = \frac{\text{mass of shell } rqp}{\text{mass of shell } PQ}.$$

Now the shell rqp can be broken up into elements of mass formed as $d\omega'$ has been formed, and the corresponding elements, $d\omega$, will completely exhaust the shell PQ; hence, taking all the elements of the inner shell, and all the corresponding elements of the outer, and thus exhausting both shells, we see that

$$\frac{\text{the Potential of the inner shell at } P}{\text{the Potential of the outer shell at } p} = \frac{\text{mass of inner shell}}{\text{mass of outer shell}}.$$

Now since these shells are bounded each by similar surfaces, the Potential of the outer shell is constant at all internal points, and (in virtue of the continuity of the Potential) this Potential is the same as the Potential of the outer shell at P.

Hence the Potential of an ellipsoidal shell bounded by similar surfaces is constant at all points on the surface of any ellipsoid confocal with the surface of the shell—that is, the level surfaces of an ellipsoidal shell are confocal ellipsoids, and its attraction at any point is therefore normal to the confocal ellipsoid through the point.

Let V and V' be the Potentials of the shells PQ and rqp at P; then

$$V' = \frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma}V. \tag{4}$$

We shall show in the next section (Example 4) that these shells produce at *all* points outside both Potentials which are proportional simply to the masses of the shells, i. e. related as in (4); so that at all such points their attraction-intensities also bear this relation to each other. Hence at any point outside both shells—even though it is just on the outer surface of PQ—we have

$$\frac{dV'}{dx} = \frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma} \cdot \frac{dV}{dx}. \tag{5}$$

For this reason the calculation of the attraction of an ellipsoidal shell at an external point is reduced to that of a shell at a point on its surface.

344.] **Attraction of an Ellipsoid at an External Point.** Let ABD (Fig. 286) be a solid homogeneous ellipsoid, and let it be required to find its attraction on a unit mass condensed at P. Break the ellipsoid up into an infinite number of thin shells bounded by ellipsoids similar to each other and to the surface ABD; let one of these shells be that between the surfaces $vr'p'$ and rqp. Denote this shell by (s); and describe the ellipsoids PQ and msn, similar to each other and confocal with the surfaces of (s), as in the preceding Articles. Denote this shell by (σ).

Let the axes of ABD be a, b, c; let those of rqp be ka, kb, kc, and let those of $vr'p'$ be $(k+dk)a$, $(k+dk)b$, $(k+dk)c$. Also, let the axes of the ellipsoid PQ be $k\sqrt{a^2+\lambda^2}$, $k\sqrt{b^2+\lambda^2}$, $k\sqrt{c^2+\lambda^2}$; then, by the last Art., those of msn will be $(k+dk)\sqrt{a^2+\lambda^2}$, $(k+dk)\sqrt{b^2+\lambda^2}$, $(k+dk)\sqrt{c^2+\lambda^2}$. Now (Art. 322), the attraction of the shell (σ) on a unit mass at P is

$$4\pi\gamma\rho \, . \, Pn,\text{*}$$

where Pn is the normal thickness of the shell at P. This attraction acts in the direction of the normal Pn, whose direction cosines are

$$\frac{px}{k^2(a^2+\lambda^2)}, \quad \frac{py}{k^2(b^2+\lambda^2)}, \quad \frac{pz}{k^2(c^2+\lambda^2)},$$

p being the length of the perpendicular from C, the centre of the ellipsoid on the tangent plane at P, and x, y, z the coordinates of P. Hence the attraction of (σ) on P parallel to the axis of x, in the positive direction, is

$$-\frac{4\pi\gamma\rho\, px}{k^2(a^2+\lambda^2)} \cdot Pn. \tag{1}$$

* The curious compensation of errors involved in the usual proof of this is well noticed by Collignon (*Dynamique*, p. 403).

Draw the line CP meeting the inner surface of (σ) in s. Then $\frac{Pn}{Ps} = \frac{p}{CP}$, therefore $Pn = p \cdot \frac{Ps}{CP}$. But $\frac{Cs}{CP} = \frac{\text{axis of } msn}{\text{axis of } PQ}$ $= \frac{k+dk}{k}$; therefore $\frac{Ps}{CP} = -\frac{dk}{k}$, and $Pn = -\frac{pdk}{k}$.

Substituting this value in (1), we find the attraction of (σ) parallel to the axis of x to be

$$\frac{4\pi\gamma\rho p^2 x dk}{k^3(a^2+\lambda^2)}.$$

Multiplying this by the ratio of the mass of (s) to that of (σ), we have the component of the attraction of (s). Denoting this latter by dX, we have

$$dX = \frac{4\pi\gamma\rho abc p^2 x dk}{k^3(a^2+\lambda^2)^{\frac{3}{2}}\sqrt{(b^2+\lambda^2)(c^2+\lambda^2)}}. \tag{2}$$

Now, by the equation of the surface PQ,

$$\frac{x^2}{a^2+\lambda^2} + \frac{y^2}{b^2+\lambda^2} + \frac{z^2}{c^2+\lambda^2} = k^2.$$

Differentiating this, regarding k and λ as variables, we have

$$\frac{k^3}{p^2}\lambda d\lambda = -dk,$$

by the well-known value of the perpendicular from the centre on the tangent plane of an ellipsoid.

Substituting this value of dk in (2), we have

$$dX = -\frac{4\pi\gamma\rho abcx\lambda d\lambda}{(a^2+\lambda^2)^{\frac{3}{2}}\sqrt{(b^2+\lambda^2)(c^2+\lambda^2)}}.$$

To find the limits of λ, we observe that when the shell (s) is taken at the centre, $k=0$; but the axes of (σ) must be finite; and as they are $k\sqrt{a^2+\lambda^2}$, &c., the value of λ corresponding to a vanishing shell at the centre is ∞. Again, if $k=1$, or (s) is a shell at the surface ABD, we have $a^2+\lambda^2 = a_1^2$, where a_1 is the semi-axis of the ellipsoid confocal with ABD, and passing through P. Denote this value of λ by λ_1. Then, if M be the mass of the solid ellipsoid ABD, we have

$$X = 3\gamma Mx\int_{\infty}^{\lambda_1}\frac{\lambda d\lambda}{\sqrt{(a^2+\lambda^2)^3(b^2+\lambda^2)(c^2+\lambda^2)}}; \tag{3}$$

and in the same way for the other components, Y and Z,

$$\left.\begin{aligned} Y &= 3\gamma My\int_{\infty}^{\lambda_1}\frac{\lambda\,d\lambda}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)^3(c^2+\lambda^2)}}, \\ Z &= 3\gamma Mz\int_{\infty}^{\lambda_1}\frac{\lambda\,d\lambda}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)^3}}. \end{aligned}\right\} \tag{4}$$

If $L=\int_{\infty}^{\lambda_1}\dfrac{\lambda\,d\lambda}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)}}$, we have evidently

$$X=-6\gamma Mx\frac{dL}{d(a^2)},\quad Y=-6\gamma My\frac{dL}{d(b^2)},\quad Z=-6\gamma Mz\frac{dL}{d(c^2)}.$$

The expressions for X, Y, Z may be put into other forms which are useful in practice, by putting

$$\lambda=\frac{c\sqrt{1-u^2}}{u}.$$

Then

$$\left.\begin{aligned} X &= -\frac{3\gamma Mx}{c^3}\int_0^{\frac{c}{c_1}}\frac{u^2\,du}{\sqrt{(1+e^2u^2)^3(1+e'^2u^2)}}, \\ Y &= -\frac{3\gamma My}{c^3}\int_0^{\frac{c}{c_1}}\frac{u^2\,du}{\sqrt{(1+e^2u^2)(1+e'^2u^2)^3}}, \\ Z &= -\frac{3\gamma Mz}{c^3}\int_0^{\frac{c}{c_1}}\frac{u^2\,du}{\sqrt{(1+e^2u^2)(1+e'^2u^2)}}, \end{aligned}\right\} \tag{5}$$

where $e^2=\dfrac{a^2-c^2}{c^2}$, and $e'^2=\dfrac{b^2-c^2}{c^2}$, the least semi-axis being c.

If the attracted particle is *on* the surface ABD of the attracting ellipsoid, the limits of u are 0 and 1, since $c_1=c$.

If the attracted point is inside the ellipsoid, let an ellipsoid be described through it concentric with and similar to the surface ABD, and the portion between these two surfaces exerts no attraction at the point (Art. 319).

Equations (5) show that the components along the principal axes of the attraction of a homogeneous ellipsoid on a particle placed anywhere on its surface or inside its mass are of the forms

$$Ax,\quad By,\quad Cz, \tag{6}$$

where A, B, C are constant quantities.

345.] **Potential of an Ellipsoid.** *Potential of a homogeneous Ellipsoid at its centre.* Let ABD (Fig. 286) be a homogeneous Ellipsoid of density ρ, whose semi-axes are a, b, c. The polar element of volume being $r^2\sin\theta\,dr\,d\theta\,d\phi$, the Potential of this

at C is $\gamma\rho r \sin\theta dr d\theta d\phi$; and integrating this from C to the bounding surface, we get $\frac{1}{2}\gamma\rho R^2 \sin\theta d\theta d\phi$, so that

$$V_0 = \tfrac{1}{2}\gamma\rho\int_0^\pi\int_0^{2\pi}\frac{\sin\theta d\theta d\phi}{\frac{\sin^2\theta\cos^2\phi}{a^2}+\frac{\sin^2\theta\sin^2\phi}{b^2}+\frac{\cos^2\theta}{c^2}},$$

where V_0 is the Potential at C. This, again, is the same as

$$V_0 = 4\gamma\rho\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\sin\theta d\theta d\phi}{\frac{\sin^2\theta\cos^2\phi}{a^2}+\frac{\sin^2\theta\sin^2\phi}{b^2}+\frac{\cos^2\theta}{c^2}},$$

Integrating with respect to ϕ, we have

$$V_0 = 2\pi\gamma\rho\int_0^{\frac{\pi}{2}}\frac{\sin\theta d\theta}{\sqrt{\left(\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{c^2}\right)\left(\frac{\sin^2\theta}{b^2}+\frac{\cos^2\theta}{c^2}\right)}}. \qquad (1)$$

Putting $\tan\theta = t$, we have

$$V_0 = 2\pi\gamma\rho abc^2\int_0^\infty\frac{t dt}{\sqrt{(1+t^2)(a^2+c^2t^2)(b^2+c^2t^2)}};$$

or, finally, putting $c^2t^2 = \lambda^2$, we have the symmetrical form

$$V_0 = 2\pi\gamma\rho abc\int_0^\infty\frac{\lambda d\lambda}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)}}. \qquad (2)$$

Potential of Ellipsoid at any internal point. Let p (Fig. 286) be the internal point the Potential at which we desire to find. Drawing the ellipsoid pqr, which is similar to the bounding surface DBA, the values of X, Y, Z at p are due entirely to the matter within pqr. Hence if the axes of pqr are ka, kb, kc, we are to put $\lambda_1 = 0$ in equations (3), (4), of the last Art., and

$$M = \tfrac{4}{3}\pi k^3\rho abc.$$

Thus we have

$$X = 4\pi\gamma\rho k^3 abcx\int_\infty^0\frac{\lambda d\lambda}{\sqrt{(k^2a^2+\lambda^2)^3(k^2b^2+\lambda^2)(k^2c^2+\lambda^2)}}.$$

Putting $\lambda^2 = k^2\mu^2$,

$$X = 4\pi\gamma\rho abcx\int_\infty^0\frac{\mu d\mu}{\sqrt{(a^2+\mu^2)^3(b^2+\mu^2)(c^2+\mu^2)}}. \qquad (3)$$

Similarly

$$Y = 4\pi\gamma\rho abcy\int_\infty^0\frac{\mu d\mu}{\sqrt{(a^2+\mu^2)(b^2+\mu^2)^3(c^2+\mu^2)}}, \qquad (4)$$

$$Z = 4\pi\gamma\rho abcz\int_\infty^0\frac{\mu d\mu}{\sqrt{(a^2+\mu^2)(b^2+\mu^2)(c^2+\mu^2)^3}}. \qquad (5)$$

Denote these values of X, Y, Z by Ax, By, Cz, the quantities A, B, C being obviously the same for all internal points. Then, if V is the Potential of the whole ellipsoid at p,

$$dV = Ax\,dx + By\,dy + Cz\,dz.$$

Integrating, we get

$$V = V_0 + \tfrac{1}{2}(Ax^2 + By^2 + Cz^2). \tag{6}$$

Substituting the values of V_0, A, B, C, just found, we have

$$V = 2\pi\gamma\rho abc \int_0^\infty \left(1 - \frac{x^2}{a^2+\mu^2} - \frac{y^2}{b^2+\mu^2} - \frac{z^2}{c^2+\mu^2}\right) \frac{\mu\,d\mu}{\sqrt{(a^2+\mu^2)(b^2+\mu^2)(c^2+\mu^2)}}. \tag{7}$$

The integrals involved in these several coefficients are easily reduced to the ordinary forms of elliptic integrals. Thus, assuming that the axes in order of descending magnitudes are a, b, c, assume

$$\mu^2 = b^2 \tan^2\phi - c^2 \sec^2\phi. \tag{8}$$

Denoting $\sqrt{(a^2+\mu^2)(b^2+\mu^2)(c^2+\mu^2)}$ by $f(\mu)$, we have

$$\frac{\mu\,d\mu}{f(\mu)} = \frac{d\phi}{\sqrt{a^2 - c^2 - (a^2-b^2)\sin^2\phi}} \tag{9}$$

$$= \frac{1}{\sqrt{a^2-c^2}} \cdot \frac{d\phi}{\Delta\phi}, \tag{10}$$

in the ordinary notation of elliptic integrals. Also

$$\left.\overline{\mu}\right|_0^\infty = \left.\overline{\phi}\right|_{\sin^{-1}\frac{c}{b}}^{\frac{\pi}{2}}.$$

Hence, denoting $\sin^{-1}\frac{c}{b}$ by ω, we have

$$V_0 = \frac{2\pi\gamma\rho abc}{\sqrt{a^2-c^2}} \int_\omega^{\frac{\pi}{2}} \frac{d\phi}{\Delta\phi}, \tag{11}$$

$$X = \frac{4\pi\gamma\rho abc}{(a^2-c^2)^{\frac{3}{2}}} \int_\omega^{\frac{\pi}{2}} \frac{\cos^2\phi\,d\phi}{\Delta^3\phi} \times x, \tag{12}$$

$$Y = \frac{4\pi\gamma\rho abc}{(b^2-c^2)\sqrt{a^2-c^2}} \int_\omega^{\frac{\pi}{2}} \frac{\cos^2\phi\,d\phi}{\Delta\phi} \times y, \tag{13}$$

$$Z = \frac{4\pi\gamma\rho abc}{(b^2-c^2)\sqrt{a^2-c^2}} \int_\omega^{\frac{\pi}{2}} \frac{\cot^2\phi\,d\phi}{\Delta\phi} \times z. \tag{14}$$

The integral in (12) is reduced to elliptic integrals of the first

and second kinds by the formula

$$\int \frac{d\phi}{\Delta^3\phi} = \frac{1}{k'^2} E(k, \phi) - \frac{k^2 \sin\phi\cos\phi}{k'^2 \Delta\phi},$$

while that in (14) is reduced thus:

$$\int \frac{\cot^2\phi\, d\phi}{\Delta\phi} = -\int \frac{d\cot\phi}{\Delta\phi} - \int \frac{d\phi}{\Delta\phi} = -\frac{\cot\phi}{\Delta\phi} - k'^2 \int \frac{d\phi}{\Delta^3\phi},$$

k' being the complement of k, i. e. $k^2 + k'^2 = 1$.

Potential of an Ellipsoidal Shell at any external point. Let it be required to find the Potential at P (Fig. 286) due to the homogeneous shell contained between pqr and $p'r'v$. By Art. 342, this Potential $= \dfrac{abc}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)}}$ times the Potential produced at C by the shell $PQms$; and since the axes of the outer surface, PQ, are $k\sqrt{a^2+\lambda^2}$, &c., it is easily seen that this latter Potential is

$$-8\gamma\rho k dk \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta\, d\phi}{\frac{\sin^2\theta\cos^2\phi}{a^2+\lambda^2} + \frac{\sin^2\theta\sin^2\phi}{b^2+\lambda^2} + \frac{\cos^2\theta}{c^2+\lambda^2}}. \qquad (15)$$

But we have shown that the double integral in this expression is equal to

$$\frac{\pi}{2} f(\lambda) \int_0^{\infty} \frac{\mu\, d\mu}{\sqrt{(a^2+\lambda^2+\mu^2)(b^2+\lambda^2+\mu^2)(c^2+\lambda^2+\mu^2)}}, \qquad (16)$$

where $f(\lambda) = \sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)}$. Hence if dV is the Potential of the shell pqr at P,

$$dV = -4\pi\gamma\rho abc\, k dk \int_0^{\infty} \frac{\mu\, d\mu}{\chi(\mu)}, \qquad (17)$$

where $\chi(\mu)$ is the denominator under the integral in (16).

We may put $\lambda^2 + \mu^2 = v^2$, so that $\mu \Big|_0^{\infty} = v \Big|_{\lambda}^{\infty}$, and

$$dV = -4\pi\gamma\rho abc\, k dk \int_{\lambda}^{\infty} \frac{v\, dv}{f(v)}.$$

But, Art. 344, $kdk = -\left[\dfrac{x^2}{(a^2+\lambda^2)^2} + \dfrac{y^2}{(b^2+\lambda^2)^2} + \dfrac{z^2}{(c^2+\lambda^2)^2}\right]\lambda d\lambda$.

Hence integrating from $\lambda = \infty$ to $\lambda = \lambda_1$, so as to include the whole given ellipsoid,

$$V = 4\pi\gamma\rho abc \int_{\lambda_1}^{\infty} \left\{\left[\frac{x^2}{(a^2+\lambda^2)^2} + \frac{y^2}{(b^2+\lambda^2)^2} + \frac{z^2}{(c^2+\lambda^2)^2}\right] \int_{\lambda}^{\infty} \frac{v\, dv}{f(v)}\right\} \lambda d\lambda. \quad (1$$

This is easily reduced to a simpler form thus. Let

$$\int \frac{u\,du}{f(u)} \equiv \phi(u); \text{ then}$$

$$V = 4\pi\gamma\rho abc \int_{\lambda_1}^{\infty} \left[\frac{x^2}{(a^2+\lambda^2)^2} + \frac{y^2}{(b^2+\lambda^2)^2} + \frac{z^2}{(c^2+\lambda^2)^2}\right] [\phi(\infty) - \phi(\lambda)]\,\lambda\,d\lambda.$$

Now taking the term in x only, we have

$$x^2 \int_{\lambda_1}^{\infty} \frac{\lambda\,d\lambda}{(a^2+\lambda^2)^2} [\phi(\infty) - \phi(\lambda)] = \frac{x^2}{a^2+\lambda_1^2} \frac{\phi(\infty) - \phi(\lambda_1)}{2} - \tfrac{1}{2} \int_{\lambda_1}^{\infty} \frac{x^2}{a^2+\lambda^2} \frac{\lambda\,d\lambda}{f(\lambda)}.$$

Adding the terms in y and z, we have, since $\frac{x^2}{a^2+\lambda_1^2} + \ldots = 1$

$$V = 2\pi\gamma\rho abc\left[\phi(\infty) - \phi(\lambda_1) - \int_{\lambda_1}^{\infty} \left(\frac{x^2}{a^2+\lambda^2} + \frac{y^2}{b^2+\lambda^2} + \frac{z^2}{c^2+\lambda^2}\right)\frac{\lambda\,d\lambda}{f(\lambda)}\right]$$

$$= 2\pi\gamma\rho abc \int_{\lambda_1}^{\infty} \left(1 - \frac{x^2}{a^2+\lambda^2} - \frac{y^2}{b^2+\lambda^2} - \frac{z^2}{c^2+\lambda^2}\right)\frac{\lambda\,d\lambda}{f(\lambda)}. \qquad (19)$$

Examples.

1. Find the attraction of a homogeneous ellipsoid of revolution round the minor axis (oblate spheroid) on a particle placed on its surface.

Here $a = b$, and $e = e'$ in equations (5), Art. 344; therefore

$$X = -\frac{3\gamma Mx}{c^3} \int_0^1 \frac{u^2\,du}{(1+e^2u^2)^2}.$$

The integral is most easily found by putting $eu = \tan\theta$. We then obtain

$$X = -\frac{3\gamma Mx}{2c^3e^3}\left(\tan^{-1}e - \frac{e}{1+e^2}\right);$$

$$Y = -\frac{3\gamma My}{2c^3e^3}\left(\tan^{-1}e - \frac{e}{1+e^2}\right);$$

$$Z = -\frac{3\gamma Mz}{c^3e^3}(e - \tan^{-1}e).$$

These expressions are of importance in the theory of the Figure of the Earth.

2. A homogeneous fluid mass, self-attracting according to the law of nature, is acted upon at every element by a force proportional to

the mass of the element and its distance from an axis passing through the centre of mass of the fluid. Prove that an ellipsoid of revolution round the axis is a possible figure of equilibrium of the fluid.

Let kr be the force emanating from the axis on a unit mass at distance r from the axis. Take the axis as axis of z, and assume the surface of the fluid to be an ellipsoid of revolution whose axes are $c\sqrt{1+e^2}$, $c\sqrt{1+e^2}$, c.

Then the x-component of force on a unit mass on the surface is $(-A+k)x$, where A has the value in Example 1. Hence if V is the potential at the surface

$$dV = (-A+k)x\,dx + (-A+k)y\,dy - Cz\,dz,$$

which is zero, since if the potential is not constant over the surface of a fluid, there will be a force in the tangent plane causing a flow from one point to another. Also by differentiating the equation of the surface, we have

$$\frac{x\,dx+y\,dy}{1+e^2}+z\,dz=0.$$

Hence we must have

$$\frac{-A+k}{C}=-\frac{1}{1+e^2}.$$

Substituting the values of A and C from last example, and putting $M=\frac{4}{3}\pi c^3(1+e^2)\rho$, where ρ is the density of the fluid, we obtain

$$\frac{ke^3}{2\pi\rho}+3e=(3+e^2)\tan^{-1}e.$$

Put $k=\frac{4}{3}\pi\rho\,.\,q$; then we have

$$\frac{2qe^3+9e}{3(3+e^2)}-\tan^{-1}e=0,$$

which determines e, the eccentricity, in terms of q; and c, the least axis, is known from M, the whole mass of the fluid.

There is a major limit to the value of q in order that equilibrium in the ellipsoidal form may be possible; but into the discussion of this, which is somewhat tedious, we do not enter.

3. If from a solid homogeneous ellipsoid any complete ellipsoid is removed, find the attraction at a point—(a) inside the remaining mass, (b) inside the ellipsoidal cavity.

The attraction is to be found by considering the cavity to be filled with matter of the same density as that of the rest, and then subtracting the results due to the matter which is imagined to fill the cavity.

Let the axes of the complete ellipsoid be taken as those of reference, and let the axes of the cavity make angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ with them. Also let the co-ordinates of the attracted particle with reference to these axes be (x, y, z) and (x', y', z'), respectively, and let the components of attraction along these sets of axes be (X, Y, Z) and (X', Y', Z').

Then
$$X = Ax, \quad Y = By, \quad Z = Cz,$$
where A, B, C are constants; and
$$X' = A'x', \quad Y' = B'y', \quad Z' = C'z',$$
where if the attracted particle is outside the cavity, A', B', C' are variables, but if inside, constants.

The whole force parallel to the axis of x on a unit particle is obviously
$$X - (X' \cos \alpha_1 + Y' \cos \alpha_2 + Z' \cos \alpha_3),$$
with similar expressions for the components along the axes of y and z.

If the attracted particle is inside the cavity, the level surface passing through it is easily found. For, the virtual work of the attraction of the whole ellipsoid is $Xdx + Ydy + Zdz$, or $\frac{1}{2}d(Ax^2 + By^2 + Cz^2)$; and that of the attraction of the small ellipsoid is $X'dx' + Y'dy' + Z'dz'$, or $\frac{1}{2}d(A'x'^2 + B'y'^2 + C'z'^2)$. Hence the level surfaces inside the cavity are given by the equation
$$Ax^2 + By^2 + Cz^2 - A'x'^2 - B'y'^2 - C'z'^2 = \text{const.}$$
They are therefore quadrics.

We could in the same way find the effect due to an ellipsoidal mass which contains in its interior another ellipsoidal mass (or nucleus) of density different from that of the remainder. If ρ and ρ' are the densities of the two portions ($\rho' > \rho$), imagine the whole to consist of a homogeneous mass of density ρ, and add the effect due to the nucleus, supposed of density $\rho' - \rho$.

4. Prove that an oblate spheroid of uniform density cannot have its own surface for one of its level surfaces.

[The condition that its own surface should be a level surface is $\tan^{-1}e = \dfrac{3e}{3+e^2}$, which cannot be satisfied by any value of e, except zero.]

5. Prove that a prolate spheroid of uniform density cannot have its own surface for a level surface.

[By putting $e = k\sqrt{-1}$ in the last result, the required condition becomes
$$\tfrac{1}{2}\log\frac{1+k}{1-k} = \frac{3k}{3-k^2};$$
which gives by expansion
$$(3-k^2)(1+\tfrac{1}{3}k^2+\tfrac{1}{5}k^4+\dots) = 3, \quad \text{or} \quad \frac{1}{3.5}+\frac{2k^2}{5.7}+\dots = 0,$$
which is, of course, quite impossible.]

6. Prove that in the spheroid considered in Example 2 the resultant attraction at any point on the surface is proportional to the length of the normal between that point and the axis of revolution.

7. Express gravity on the surface of such a spheroid in terms of the latitude.

[The latitude of a point on the surface is the angle made with the plane of the equator by the normal at the point.

If E denotes the value of gravity at the equator, G the value in latitude λ, and ϵ the eccentricity of the generating ellipse,

$$G = \frac{E}{\sqrt{1-\epsilon^2 \sin^2\lambda}};$$

so that if ϵ is small, the increase of gravity at any point above the equatorial value is proportional to $\sin^2$ (latitude).]

8. The components of attraction of a homogeneous ellipsoid at an internal point (x, y, z) being Ax, By, Cz, prove that

$$A+B+C = -4\pi\gamma\rho,$$

where ρ is the density at the point.

9. From a continuous mass, M, a portion M' is removed and reduced to a state of infinite diffusion; show that the work thus done is

$$\int V dm' - \tfrac{1}{2}\int V' dm',$$

the integrals being extended throughout the volume of M' (while it forms part of M), V being the Potential at any point of M' due to the complete mass, V' the Potential due to M' alone, and dm' an element of M'.

10. A homogeneous ellipsoid of density ρ and semi-axes a, b, c, contains a concentric spherical cavity of radius r; prove that the work done in filling the cavity with homogeneous matter of density ρ, brought from a state of diffusion, is

$$\frac{8}{15}\gamma\pi^2\rho^2 r^3\left\{5\, abc\int_0^\infty \frac{\lambda\, d\lambda}{f(\lambda)} - 3\, r^2\right\},$$

where $f(\lambda) \equiv \sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)}$, and verify this result for the case in which the ellipsoid is a sphere (Example 23, Art. 332).

(Use the value of V in (6), Art. 345, and observe that

$$A+B+C = -4\pi\gamma\rho,$$

and also that $\int x^2 dm' = \int y^2 dm' = \int z^2 dm' = \frac{4}{15}\pi\rho r^5$.)

11. If the external level surfaces of any attracting system are confocal ellipsoids defined by the parameter λ in the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1,$$

show that the potential is given by the equation

$$V = \gamma\frac{M}{2}\int\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

where M is the mass of the attracting system.

(Transform the equation $\nabla^2 V = 0$, which holds for all points outside the mass, into a differential equation in which λ is the independent variable. Thus

$$\nabla^2 V = \frac{d^2 V}{d\lambda^2}\left\{\left(\frac{d\lambda}{dx}\right)^2 + \left(\frac{d\lambda}{dy}\right)^2 + \left(\frac{d\lambda}{dz}\right)^2\right\} + \frac{dV}{d\lambda}\nabla^2\lambda = 0.$$

But if p is the central perpendicular on the tangent plane to the ellipsoid at any point, we easily find by differentiating the equation connecting λ with x, y, z, that

$$\frac{d\lambda}{dx} = \frac{2p^2 x}{a^2+\lambda}, \text{ and that } \left(\frac{d\lambda}{dx}\right)^2 + \ldots = 4p^2,$$

while $$\nabla^2\lambda = 2p^2\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right).$$ Therefore, &c.)

SECTION IV.—*Green's Equation and Spherical Harmonics.*

346.] Green's Equation. Let U and V be any finite and continuous functions of the co-ordinates of a point in space, and let ∇^2 stand, as usual, for the operation $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$.

Take any closed surface (Fig. 287, or Fig. 274, Art. 316); let $d\Omega$ represent an element of volume of the space inside this surface, and let dS represent any element of area of the surface. Then we shall have the following equation, which is due to Green:

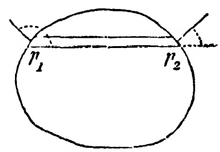

Fig. 287.

$$\int U\nabla^2 V d\Omega = \int U\frac{dV}{dn}dS - \int\left(\frac{dU}{dx}\frac{dV}{dx}+\frac{dU}{dy}\frac{dV}{dy}+\frac{dU}{dz}\frac{dV}{dz}\right)d\Omega, \quad (\alpha)$$

dn in this equation being an element of the normal to the surface drawn outwards, as in Art. 329.

For $d\Omega = dxdydz$, so that the left-hand side is

$$\iiint U\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)dxdydz.$$

Consider the term $U\frac{d^2V}{dx^2}$ separately. Taking $\int U\frac{d^2V}{dx^2}dx$, and integrating between the extreme values of x, considering y and z both constant—i.e. in the figure, performing a summation along the line p_1p_2 parallel to the axis of x—we get

$$\int U\frac{d^2V}{dx^2}dx = \left(U\frac{dV}{dx}\right)_2 - \left(U\frac{dV}{dx}\right)_1 - \int\frac{dU}{dx}\frac{dV}{dx}dx, \quad (1)$$

in which the suffixes denote the values of the quantities in brackets at the extreme points p_2 and p_1 of the integration. Fig. 287 represents the line p_2p_1 as meeting the given closed

surface in only two points, but our result holds whatever be the number of these points—observing that it must be an *even* number, since the surface is closed. Fig. 274 will represent the more general case if we imagine the line OQ to be parallel to the axis of x; and with this figure the terms outside the sign of integration in (1) would be

$$\left(U\frac{dV}{dx}\right)_2 - \left(U\frac{dV}{dx}\right)_1 + \left(U\frac{dV}{dx}\right)_4 - \left(U\frac{dV}{dx}\right)_3. \tag{2}$$

The integration along $p_1 p_2$ is performed in reality along a very slender parallelepiped whose transverse section is $dy\,dz$, and not along a line. Multiplying the different terms of (1) by $dy\,dz$, we have the right-hand side equal to

$$\left(U\frac{dV}{dx}\right)_2 dy\,dz - \left(U\frac{dV}{dx}\right)_1 dy\,dz - dy\,dz\int\frac{dU}{dx}\frac{dV}{dx}dx. \tag{3}$$

Now if dS_2 is the element of surface cut off by the parallelepiped at p_2 and if λ_2 is the angle (represented by the dotted line) made with the axis of x by the outward-drawn normal at p_2, we have

$$dy\,dz = \cos\lambda_2 \,.\, dS_2; \tag{4}$$

and if dS_1 is the element of area cut off at p_1, while λ_1 is the direction angle of the outward-drawn normal at p_1, *measured in the same sense as at* p_2, we have

$$dy\,dz = -\cos\lambda_1 \,.\, dS_1; \tag{5}$$

with exactly like results in the general figure, Fig. 274, the cosines being negative at the points P_1, P_3, and positive at P_2, P_4. Hence (3) becomes

$$\left(U\frac{dV}{dx}\cos\lambda\, dS\right)_2 + \left(U\frac{dV}{dx}\cos\lambda\, dS\right)_1 - dy\,dz\int\frac{dU}{dx}\frac{dV}{dx}dx; \tag{6}$$

and hence

$$\int U\frac{d^2V}{dx^2}d\Omega = \int U\frac{dV}{dx}\cos\lambda\, dS - \iiint\frac{dU}{dx}\frac{dV}{dx}dx\,dy\,dz, \tag{7}$$

λ denoting the angle made by the normal at any point with the axis of x.

In the same way, if μ and ν are the angles made by the normal with the axes of y and z, we have

$$\int U\frac{d^2V}{dy^2}d\Omega = \int U\frac{dV}{dy}\cos\mu\, dS - \iiint\frac{dU}{dy}\frac{dV}{dy}dx\,dy\,dz, \tag{8}$$

$$\int U\frac{d^2V}{dz^2}d\Omega = \int U\frac{dV}{dz}\cos\nu\, dS - \iiint\frac{dU}{dz}\frac{dV}{dz}dx\,dy\,dz. \tag{9}$$

Adding (7), (8), and (9) together, we obtain the equation (α).

Writing down the value of $\int V\nabla^2 U d\Omega$, and subtracting the result from (α), we obtain

$$\int(U\nabla^2 V - V\nabla^2 U)d\Omega = \int\left(U\frac{dV}{dn} - V\frac{dU}{dn}\right)dS. \qquad (\beta)$$

If U is taken identical with V, we have the result

$$\int V\nabla^2 V d\Omega = \int V\frac{dV}{dn}dS - \int\left[\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right]d\Omega. \quad (\gamma)$$

Green's Equation is probably the most remarkable and powerful analytical result in the whole range of Mathematical Physics. It is put by Sir W. Thomson into the following somewhat generalized form

$$\int U\left(\frac{d\,.\,\phi V_1}{dx} + \frac{d\,.\,\phi V_2}{dy} + \frac{d\,.\,\phi V_3}{dz}\right)d\Omega$$

$$= \int U\phi\frac{dV}{dn}dS - \int\phi(U_1V_1 + U_2V_2 + U_3V_3)d\Omega, \quad (\delta)$$

where ϕ is any function whatever and

$$U_1 = \frac{dU}{dx},\ V_1 = \frac{dV}{dx}, \text{ \&c.}$$

This is at once deducible as before.

Green's equation holds also for the space included between any closed surface S (Fig. 288) and any closed surfaces, M_1, M_2, included by S. In this case the boundary of the space considered is not continuous—that is, starting from any one point, P_1, on the boundary, it is not possible to reach every other point (such as P_4) on the boundary by travelling merely over the boundary itself.

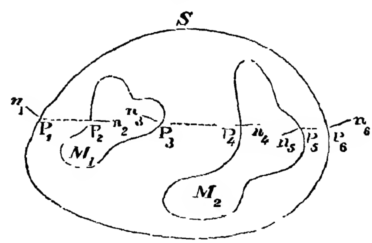

Fig. 288.

The figure represents a line $P_1 P_2 \ldots P_6$ parallel to the axis of x along which the integration $\int U\frac{d^2V}{dx^2}dx$ is performed, and the lines P_1n_1, P_2n_2, P_3n_3, ... are the elements of the normals drawn outwards from the space considered, i.e. the space included between the contours of S, M_1, and M_2.

The functions U and V may be any whatever—subject to the conditions of being finite, continuous, and (as we shall assume for the present) single-valued.

Take $U = C =$ any constant, for example, and the equation (α) becomes

$$\int \nabla^2 V d\Omega \int \frac{dV}{dn} dS. \tag{10}$$

If V is the Potential due to any attracting matter,

$$\nabla^2 V = -4\pi\rho\gamma$$

Art. 329, and we have at once the equation

$$-4\pi\gamma M_i = \int N dS, \tag{11}$$

as in Art. 324; and if the surface S has all the attracting matter outside it, we have in the same way $\int N dS = 0$ (Art. 324); for in Fig. 288 let the contour of M_2 represent any closed surface, let M_1 represent any attracting matter outside this surface, and let S be any surface completely surrounding both. Applying Green's equation (10) to the space included between the surfaces S and M_2 and the contours of these surfaces, we have

$$-4\pi\gamma M_1 = \int N dS + \int N' dS',$$

where N and dS refer to the surface S, and N' and dS' to the surface of M_2. But, ignoring the surface of M_2 altogether, (11) gives

$$-4\pi\gamma M_1 = \int N dS.$$

Hence $\int N' dS' = 0$.

The quantity $-\nabla^2 V$ is called by Clerk Maxwell the *concentration* of V. Hence (10) asserts that, if a function has no concentration at any point inside a closed surface, the surface-integral of the normal variation of this function over the surface is zero.

As another example, take $U = V$, and let V be the Potential due to any attracting matter. Then the equation becomes

$$4\pi\gamma \int V dm = -\int V \frac{dV}{dn} dS + \int R^2 d\Omega, \tag{ϵ}$$

where R is the resultant force intensity at any point inside the closed surface, $dm = \rho d\Omega =$ element of mass at any point inside, and γ, as usual, the gravitation constant. Now (Art. 331) the left-hand side of (ϵ) is $8\pi\gamma$ times the Potential Work of the attractive forces of the system, or, in other words, $8\pi\gamma$ times the amount of work done by these forces in bringing the system

from a state of infinite diffusion to its present configuration. Hence the right-hand side is another expression for the same thing. A simpler expression is obtained by taking the closed surface S of infinite size, i.e. every point of it at infinity. Now if none of the attracting matter is infinitely distant, $V = 0$ at every point of this infinitely distant surface; nevertheless the integral $\int \frac{dV}{dn} dS$ is finite and $= -4\pi\gamma M$, where M is the quantity of the given matter. Hence $\int V \frac{dV}{dn} dS$ over this surface must be zero, and we have

$$4\pi\gamma \int V dm = \int R^2 d\Omega, \qquad (\zeta)$$

the integral on the right-hand side being taken all through the attracting matter and through infinite space outside the attracting matter, and the work required to reduce the given self-attracting system to a state of infinite diffusion is

$$\frac{1}{8\pi\gamma}\int R^2 d\Omega, \qquad (\eta)$$

the integration extending *through all space outside the matter, and through the matter itself.*

Examples.

1. Take the case of a homogeneous solid sphere of radius a. Then at any point inside $V = 2\pi\gamma\rho(a^2 - \frac{1}{3}r^2)$, r being the distance of the point from the centre. We may take $dm = 4\pi\rho r^2 dr$, and we find $\frac{1}{2}\int V dm = \frac{3}{5}\gamma \frac{M^2}{a}$, where M = mass of sphere.

At any external point $R = \frac{\gamma M}{r^2}$; therefore

$$\int R^2 d\Omega = \gamma^2 M^2 \int\int\int \frac{1}{r^2} dr\, d\mu\, d\phi = 4\pi\gamma^2 M^2 . \frac{1}{a}.$$

At any internal point $R = -\frac{4}{3}\pi\gamma\rho r$, $\therefore \int R^2 d\Omega = \frac{4}{5}\pi\gamma^2 M^2 . \frac{1}{a}$; and the sum of these two integrals divided by $8\pi\gamma$ gives the same value of the Potential work as before. (See p. 205).

2. Supposing that a sphere of water is brought together by mutual attractions of particles from a state of infinite diffusion, find its radius if the amount of work done by these forces is sufficient to raise its temperature 1° C.

Let a centimetres be its radius. Then the number of ergs done by the forces is $\frac{3}{5}\gamma M^2/a$, where M = its mass in grammes $= \frac{4}{3}\pi a^3$.

But 1 water-gramme-centigrade degree is equivalent to 42×10^6 ergs (Joule's *Dynamical Equivalent* of heat). Hence the heat, in ergs, required to raise M grammes through $1°$ is $42 \times 10^6 \times M$. Therefore

$$\tfrac{3}{5}\gamma M^2/a = 42 \times 10^6 \times M;$$

and we know that $\gamma = 1/(1543 \times 10^4)$ dynes (Art. 321);

$$\therefore \quad a = \tfrac{1}{2}\sqrt{210 \times 1543/\pi} \times 10^5 \text{ centimetres}$$
$$= 16 \times 10^6 \text{ (roughly)}.$$

Now the Earth's radius $= 637 \times 10^6$ cms.; therefore the diameter of the required water sphere $= \frac{1}{40}$ (Earth's diameter), roughly.

3. If any surface S, enclosing a given distribution of mass, is a surface of zero potential for this mass, the potential of the system is constantly zero at all points outside S.

Draw an infinitely distant sphere enclosing the system, and apply Green's equation, taking $U = V$, to the space between this sphere and the given surface S. The volume-integral $\int V\nabla^2 V \,.\, d\Omega$ taken through this space is zero, since $\nabla^2 V$ is everywhere zero. Also the two surface-integrals $\int V \frac{dV}{dn} dS$, one taken over S and the other over the infinitely distant sphere, both vanish—the former evidently, the latter because V is of the order dm/r while dV/dn is of the order dm/r^2, and dS is of the type $r^2 d\mu d\phi$, so that the infinitely great value of r reduces to zero each term $VdSdV/dn$. Hence Green's equation reduces to $\int R^2 d\Omega = 0$, where, as in (ϵ), last Art., R is the resultant force-intensity at any point in the space considered; $\therefore R = 0$ at each point, i.e. V is constant, and equal to zero everywhere.

4. If for each of two different material systems, M and M', a certain surface, S, which encloses both, is a surface of constant potential, all the external level surfaces of M are also level surfaces of M'.

For, let A be the constant value of the potential of M on S, and let A' be the constant value for M' on S. Then, if we increase the density at every point of M' in the constant ratio A/A', we obtain a mass system occupying the position of M', whose total quantity is $M'A/A'$ and whose potential on S is A. Reverse the sign of every element of this new mass, and take this reversed system conjointly with M. We then have a mass system, $M - M'A/A'$, producing constant zero potential over the surface S, and therefore at every point outside S, by last example. Hence every level surface of M between S and infinity is also a level surface of M', and the ratio of the potentials is, on all, A/A'.

Thus is proved the equivalence of the ellipsoidal shell, qp', Fig. 286, with the shell Qs so far as attraction at all points outside both, or at the outer surface of the latter, is concerned.

5. If two different masses of equal amounts have the same external level surfaces, prove that $\iiint \rho\, U dx dy dz$ is the same for both, where U is any function satisfying Laplace's equation.

By Example 16, p. 204, we see that their Potentials must be identical at all external points. Let V be the Potential on any common level surface. Then applying Green's equation (β), p. 229, to the volume and surface of this level surface, we have for one of the masses

$$-4\pi\gamma\int U\rho\, d\Omega = \int U\frac{dV}{dn}\, dS - V\int\frac{dU}{dn}\, dS.$$

Now since U has no concentration inside the surface (p. 230) we have $\int\frac{dU}{dn}\, dS = 0$; also $\rho\, d\Omega = dm =$ the element of mass; therefore

$$\int U dm = -\frac{1}{4\pi\gamma}\int U\frac{dV}{dn}\, dS.$$

For the other mass dV/dn is the same as for the first, since their Potentials are equal at all points. Hence for it

$$\int U\, dm' = -\frac{1}{4\pi\gamma}\int U\frac{dV}{dn}\, dS,$$

which gives $\int U\, dm = \int U dm'$, as required.

If the two masses are not equal, these integrals are proportional to their amounts; or $\qquad \int U dm : M = \int U dm' : M' \qquad (\alpha)$

6. If two different masses have the same external level surfaces, they have the same centre of mass and the same principal axes at this point, and their Ellipsoids of gyration are confocal.

For, let $U = x$ in (α), and we have $\bar{x} = \bar{x}'$. Similarly if $U = y$, and $U = z$, we obtain $\bar{y} = \bar{y}'$, &c. We may take the centre of mass as origin.

Secondly, let $U = xy$ (which satisfies $\nabla^2 U = 0$); then

$$M^{-1}\int xy\, dm = M'^{-1}\int xy\, dm';$$

so that if the products of inertia round the axes of co-ordinates vanish for the first mass, they also vanish for the second. Take the principal axes as axes of co-ordinates.

Thirdly, let $U = y^2 + z^2 - 2x^2$, and if $A, \ldots A', \ldots$ are the principal moments of inertia, we have

$$(B + C - 2A)/M = (B' + C' - 2A')/M'.$$

Two similar equations also follow. Hence

$$A/M = \lambda + A'/M'; \; B/M = \lambda + B'/M'; \; C/M = \lambda + C'/M'.$$

7. Prove that the mean value of any continuous function, ϕ, taken over a sphere of radius a exceeds the value which the function has at the centre of the sphere by

$$\frac{1}{4\pi}\int\left(\frac{1}{r} - \frac{1}{a}\right)\nabla^2\phi\, d\Omega,$$

this integral being taken through the volume of the sphere, and r being the distance of any point from the centre.

Round the centre of the sphere describe a circle of extremely small

radius, b, and apply Green's equation to the space between the two spheres. This space has for boundary the surfaces of the two spheres. Let $\frac{1}{r} - \frac{1}{a}$ be taken as U. Then from (β), p. 229, since $\nabla^2 U = 0$,

$$\int\left(\frac{1}{r} - \frac{1}{a}\right)\nabla^2\phi \,.\, d\Omega = \int_1\left(\frac{1}{r} - \frac{1}{a}\right)\frac{d\phi}{dr}\,dS - \int_2\left(\frac{1}{r} - \frac{1}{a}\right)\frac{d\phi}{dr}\,dS$$
$$-\int\phi\,\frac{d\frac{1}{r}}{dr}\,dS + \int_2\phi\,\frac{d\frac{1}{r}}{dr}\,dS,$$

the integrals with suffix 1 referring to the surface of the outer sphere (for which $dn = dr$), and those with suffix 2 to the surface of the inner (for which $dn = -dr$). Now the first integral on the right-hand side is zero, $\because\ r = a$; the third integral $= \frac{1}{a^2}\int\phi dS$; the fourth $= -\frac{1}{b^2}\int\phi dS = -4\pi\phi_0$ (where ϕ_0 is the value of ϕ at the centre) because ϕ at every point on the surface of the small sphere is very nearly constant, and $\int dS = 4\pi b^2$. Also the second integral is zero, because $d\phi/dr$ is very nearly the same at all points on the small sphere, and $r = b$ at all points, so that this integral

$$= -\left(\frac{1}{b} - \frac{1}{a}\right)4\pi b^2\left(\frac{d\phi}{dr}\right)_0,$$

which is infinitely small since b is so. Hence we have

$$\int\left(\frac{1}{r} - \frac{1}{a}\right)\nabla^2\phi\, d\Omega = \frac{1}{a^2}\int\phi\, dS - 4\pi\phi_0,$$

which gives the desired result.

If ϕ is the Potential of matter wholly external to the sphere, we have the result in Example 12, p. 201.

If there is matter internal as well as external to the sphere, it can be shown at once that the mean value of the Potential on the surface is equal to the Potential at the centre due to the external mass, plus the Potential which would be produced at the centre by distributing the internal mass as a shell over the surface; in other words,

$$\frac{1}{4\pi a^2}\int V dS = V_0^{(e)} + \gamma\frac{M^{(i)}}{a}.$$

8. If ϕ is any function of the co-ordinates of a point, P, and round P as centre a small sphere, of radius r, be described, prove that if $\bar{\phi}$ is the mean value of ϕ (i.e. mean volume-value) for all points within the sphere,

$$\bar{\phi} = \phi + \tfrac{1}{10}r^2\nabla^2\phi.$$

347.] **Remarkable Consequence of Green's Equation.** The first result that we shall deduce from Green's Equation is the

following, which is of fundamental importance in the theory of Attraction—

There cannot be two different functions which both satisfy Laplace's equation at every point of a closed region of space and which have both the same value at every *point of the surface or surfaces bounding this region.*

If possible, let there be two different functions V and U such that at every point in the region enclosed by the surface in Fig. 287, or at every point in the region included between the surfaces of S, M_1, and M_2 in Fig. 288, we have

$$\nabla^2 V = 0 \text{ and } \nabla^2 U = 0,$$

and also such that $V = U$ at *every* point on the bounding surface in Fig. 287, and at every point on the surface S, every point on the surface of M_1, and every point on the surface of M_2 in Fig. 288.

Then our theorem is that V and U must be identical.

For, by Green's equation, if ϕ is *any* function,

$$\int \phi \nabla^2 \phi \, d\Omega = \int \phi \frac{d\phi}{dn} dS - \int \Big[\Big(\frac{d\phi}{dx}\Big)^2 + \Big(\frac{d\phi}{dy}\Big)^2 + \Big(\frac{d\phi}{dz}\Big)^2\Big] d\Omega, \quad (1)$$

in which $d\Omega$ is any element of volume of the space considered and dS an element of area of the boundary.

Let $\phi \equiv V - U$. Then by hypothesis $\nabla^2 \phi = 0$ at every point in the volume, and $\phi = 0$ at every point on the boundary; hence (1) becomes

$$\int [(d\phi/dx)^2 + (d\phi/dy)^2 + (d\phi/dz)^2] d\Omega = 0.$$

Now this asserts that a summation of a sum of squares is zero, which cannot be unless every term in the summation $= 0$. Hence at *every* point in the volume considered we must have

$$dV/dx = dU/dx; \; dV/dy = dU/dy; \; dV/dz = dU/dz;$$

and these require $V \equiv U$ at every point of the included space.

The application of this result to the theory of Potential is obvious. M_1, M_2 may be any distributions of attracting matter and S an infinitely distant surface. If no portion of the attracting matter is contemplated as at infinity, the Potential has a zero value at every point on S. Then the Theorem just proved, when applied to the region included between the infinitely distant surface and the contours of M_1 and M_2—i. e. to the whole of the space external to the masses M_1 and M_2—comes to this: if we

know any function V of the co-ordinates (x, y, z), which vanishes for all points at infinity, which at every point on the contours of M_1 and M_2 has the value of the Potential of these masses at the point, and which at every point outside these masses satisfies Laplace's equation $\nabla^2 V = 0$; then V is the Potential produced by the masses at any point (x, y, z) of the space external to them.

For the Potential satisfies all these conditions, and as there is only one function which can do so, the given function, V, must be the Potential.

348.] **Central Solid of Revolution. Theorem of Legendre.** For the case in which the attracting matter forms any central solid of revolution we shall now prove the following remarkable result which was first proved by Legendre: *If in the case of any body which is symmetrical, both as to shape and to density, about an axis, we know a Potential function (of x, y, z or any other co-ordinates which determine the position of a point) which for all points on the axis outside the body is the Potential of the body at these points, this function is the Potential at every point outside the body.*

[The expression 'Potential function' is here used for brevity to signify one satisfying Laplace's equation, $\nabla^2\phi = 0$.]

Legendre's proof of this theorem (which is that commonly employed) will be subsequently given. The following seems to be more simple and elementary.

The Potential for this case must be simply a function of the two cylindrical co-ordinates z, ζ (Art. 329). Hence if V is the Potential at any point,

$$d^2 V/dz^2 + d^2 V/d\zeta^2 + \zeta^{-1} dV/d\zeta = 0. \qquad (1)$$

Let U be the function which we know, and which satisfies the conditions above enunciated. Then U also satisfies (1). Let $\phi = V - U$; then ϕ also satisfies the equation (1), or

$$\zeta(d^2\phi/dz^2 + d^2\phi/d\zeta^2) + d\phi/d\zeta = 0. \qquad (2)$$

Now all along the axis of z we have $\phi = 0$, and therefore

$$d\phi/dz = 0,\ d^2\phi/dz^2 = 0,\ d^3\phi/dz^3 = 0.$$

With these conditions, and with the condition that (2) holds for all values of z and ζ, we wish to show that all the differential coefficients of ϕ, such as $d^{m+n}\phi/dz^m d\zeta^n$, vanish at all points on the axis of z.

Firstly, at all points on the axis of z (since $\zeta = 0$) we have, by (2), $d\phi/d\zeta = 0$.

Again, differentiating (2) with respect to ζ and putting $\zeta = 0$, we have $d^2\phi/d\zeta^2 = 0$.

Differentiating (2) n times with respect to ζ, we have by Leibnitz's Theorem

$$\zeta\left(\frac{d^{n+2}\phi}{dz^2 d\zeta^n} + \frac{d^{n+2}\phi}{d^{n+2}\zeta}\right) + n\frac{d^{n+1}\phi}{dz^2 d\zeta^{n-1}} + (n+1)\frac{d^{n+1}\phi}{d\zeta^{n+1}} = 0,$$

so that at all points on the axis of z

$$n\frac{d^2}{dz^2}\left(\frac{d^{n-1}\phi}{d\zeta^{n-1}}\right) + (n+1)\frac{d^{n+1}\phi}{d\zeta^{n+1}} = 0. \qquad (3)$$

Hence if at all points on the axis $\dfrac{d^{n-1}\phi}{d\zeta^{n-1}} = 0$, we shall have $\dfrac{d^{n+1}\phi}{d\zeta^{n+1}} = 0$. But $\dfrac{d\phi}{d\zeta}$ and $\dfrac{d^2\phi}{d\zeta^2}$ have both been proved to vanish at all points on the axis, and therefore all the differential coefficients of ϕ with respect to ζ vanish on the axis; and hence also, on account of the independence of the order of differentiation, all of the form $d^{m+n}\phi/dz^m d\zeta^n$ also vanish on the axis.

Now, by Maclaurin's Theorem, if $\phi = f(z, \zeta)$, we have

$$\phi = \phi_0 + z\left(\frac{d\phi}{dz}\right)_0 + \zeta\left(\frac{d\phi}{d\zeta}\right)_0 + \frac{1}{1\,.\,2}\left\{z^2\left(\frac{d^2\phi}{dz^2}\right)_0 + 2z\zeta\left(\frac{d^2\phi}{dz\,d\zeta}\right)_0 + \zeta^2\left(\frac{d^2\phi}{d\zeta^2}\right)_0\right\} + \dots$$

where ϕ_0, $\left(\dfrac{d\phi}{dz}\right)_0$, ... mean the values of ϕ and its differential coefficients when $(0, 0)$ are put for (z, ζ). Hence, by what has just been proved, ϕ is zero everywhere—that is, V is identical with U.

The same proof shows that if we know a Potential function, U, which at every point inside the attracting mass satisfies the equation $\nabla^2 U = -4\pi\gamma\rho$, and if U for all points on the axis of symmetry is the Potential, it is the Potential for *all* points in the mass. For, putting $\phi \equiv V - U$, we have still the equation (2), with all its consequences, for ϕ; and, as before, we prove $\phi = 0$ for all points.

349.] **Laplacians.** Let O be any fixed origin, P a point whose polar co-ordinates are (r, θ, ϕ) and P' a point whose co-ordinates are (r', θ', ϕ'). Then, denoting, as previously, $\cos\theta$ by μ and $\cos\theta'$ by μ', the reciprocal of the distance between P and P' is

$$1/\sqrt{r^2 - 2rr'\{\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')\} + r'^2}. \quad (\alpha)$$

Now since the reciprocal of the distance between P and any other point is the type of a Potential function ($\frac{m}{PP'}$ is, in fact, the Potential at P due to a mass m condensed at P'), it follows that the expression (α) satisfies the equation $\nabla^2(1/PP') = 0$, where $\nabla^2 \equiv d^2/dx^2 + d^2/dy^2 + d^2/dz^2$, or its equivalent operation in (r, μ, ϕ), or in (z, ζ, ϕ); and again that $\nabla'^2(1/PP') = 0$, where ∇'^2 signifies the same operations with reference to the co-ordinates of P'.

Again, the expression (α) may be developed in an infinite series proceeding by powers of the ratio r'/r, or of the ratio r/r', the coefficients of these successive powers being functions of μ, μ', ϕ, ϕ'. Moreover, *any one coefficient*—as, for instance, that of $(r'/r)^i$—is *a rational integral function of* μ, $\sqrt{1-\mu^2}\cos\phi$, $\sqrt{1-\mu^2}\sin\phi$, *and is the very same function of* μ', $\sqrt{1-\mu'^2}\cos\phi'$, $\sqrt{1-\mu'^2}\sin\phi'$. It is, again, obviously the same whether $1/PP'$ is developed in powers of r/r' or of r'/r.

Let this development be

$$1/PP' = r^{-1}\{L_0 + L_1 r'/r + L_2 r'^2/r^2 + \ldots + L_i r'^i/r^i + \ldots\}. \quad (\beta)$$

The coefficients of this development possess very remarkable properties, and we shall call them *Laplacians*, after Laplace, to whom their employment is due.

Thus L_i is the Laplacian of the i^{th} degree. We may speak of it as the Laplacian of the i^{th} degree for the two points P, P', whose angular co-ordinates are involved in it.

If, regarding r', μ', ϕ' as constant, we perform the operation ∇^2 on the right-hand side of (β), since the result is zero for all values of r and r', the coefficients of the several powers must all separately vanish. Thus we must have $\nabla^2 L_i/r^{i+1} = 0$. $\quad(\gamma)$

Similarly, if, regarding r, μ, ϕ as constant, we perform the operation ∇'^2, we must have $\nabla'^2(r'^i L_i) = 0$, and therefore of course, by symmetry,

$$\nabla^2(r^i L_i) = 0. \quad (\delta)$$

Substituting $\frac{L_i}{r^{i+1}}$ for V in (δ), p. 176, we have the differential equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dL_i}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2L_i}{d\phi^2} + i(i+1)L_i = 0, \quad (\epsilon)$$

and the substitution of $r^i L_i$ for V gives exactly the same equation.

The value of L_i can, of course, be found by simple binomial expansion of (α); but such a method is very tedious, and we shall adopt a different one.

Let λ be put for $\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')$, and let

$$(1-2\lambda h + h^2)^{\frac{1}{2}} = 1 - xh. \tag{1}$$

This gives

$$x = \lambda + h\frac{x^2-1}{2},$$

from which we can expand x in ascending powers of h by Lagrange's Theorem (Williamson's *Diff. Cal.*, Chap. VII).

Thus

$$x = \lambda + \frac{h}{1}\frac{\lambda^2-1}{2} + \frac{h^2}{1.2}\frac{d}{d\lambda}\left(\frac{\lambda^2-1}{2}\right)^2 + \dots$$

$$+ \frac{h^i}{\underline{|i}}\frac{d^{i-1}}{d\lambda^{i-1}}\left(\frac{\lambda^2-1}{2}\right)^i + \dots. \tag{2}$$

Now from (1) we have $\dfrac{dx}{d\lambda} = (1-2\lambda h + h^2)^{-\frac{1}{2}}$, which by hypothesis, when expanded in powers of h is

$$L_0 + L_1 h + L_2 h^2 + \dots + L_i h^i + \dots.$$

Differentiating (2) with respect to λ, and identifying the coefficients of h^i in the two values of $\dfrac{dx}{d\lambda}$, we have

$$L_i = \frac{1}{2^i\,\underline{|i}}\frac{d^i(\lambda^2-1)^i}{d\lambda^i}. \tag{ζ}$$

By actually expanding $(\lambda^2-1)^i$ and differentiating, we have

$$L_i = \frac{1}{2^i\,\underline{|i}}\left[2i(2i-1)\dots(i+1)\lambda^i - \&c.\right],$$

which shows that L_i is a rational integral function of the i^{th} degree of μ, $\sqrt{1-\mu^2}\cos\phi$, $\sqrt{1-\mu^2}\sin\phi$, and the very same function of μ',

$$\sqrt{1-\mu'^2}\cos\phi',\ \sqrt{1-\mu'^2}\sin\phi'.$$

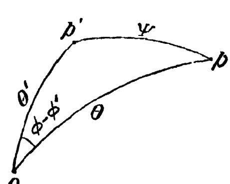

Fig. 289.

In the figure (Fig. 289) let the spherical triangle be that in which a sphere is intersected by the axis of z (from which θ and θ' are measured), and the lines OP and $O'P'$; these lines meeting the surface in o, p, p', respectively. The point o being the pole from which angles are measured, the

function $1/PP'$ satisfies the differential equation

$$\left[\frac{d}{dr}\cdot r^2\frac{d}{dr}+\frac{d}{d\mu}\cdot(1-\mu^2)\frac{d}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2}{d\phi^2}\right]\frac{1}{PP'}=0\,;$$

but, if p' is taken as pole, the expression for PP' involves only OP, OP', and $\cos\psi$ (or λ), without any term in longitude. Hence we have

$$\left[\frac{d}{dr}\cdot r^2\frac{d}{dr}+\frac{d}{d\lambda}\cdot(1-\lambda^2)\frac{d}{d\lambda}\right]\frac{1}{PP'}=0\,;$$

and putting here for $1/PP'$ the development (β), and equating to zero the coefficients of the several powers of r, we have

$$\frac{d}{d\lambda}\left\{(1-\lambda^2)\frac{dL_i}{d\lambda}\right\}+i(i+1)\,L_i=0. \qquad (\eta)$$

L_i being given by (ζ), and satisfying (η), we conclude generally that any function, X, of the form

$$X=a\frac{d^i(x^2-1)^i}{dx^i}, \qquad (3)$$

where a does not involve x, will satisfy the equation

$$\frac{d}{dx}\left\{(1-x^2)\frac{dX}{dx}\right\}+i(i+1)\,X=0. \qquad (4)$$

The value of L_i as given by (ζ) is not of much practical use. To make it useful, it must be exhibited as a series of cosines of multiples of $\phi-\phi'$ (which we may denote by ω) thus:

$$L_i=M_0+M_1\cos\omega+M_2\cos 2\omega+\ldots+M_i\cos i\omega, \qquad (5)$$

the series ending with $\cos i\omega$, because the highest power of $\cos\omega$ in L_i is the i^{th}, and we know by elementary Trigonometry that

$$2^{i-1}\cos^i\omega=\cos i\omega+i\cos(i-2)\,\omega+\frac{i\,(i-1)}{1\,.\,2}\cos(i-4)\,\omega+\ldots.$$

Laplace deduces L_i in the desired form (5) by elementary algebraic processes; but, as we prefer to present it in a more succinct form than that given by Laplace, we shall turn in the next Article to the consideration of functions, generally, which satisfy the equation $\nabla^2 V=0$.

It is to be observed that if $\mu=\mu'$ and $\phi=\phi'$, the points p and p' (Fig. 289) coincide, and $PP'=r-r'$, so that every Laplacian becomes equal to unity—as is verified by putting $\lambda=1$ in (ζ).

In general, any Laplacian for two points, p and p', has reference to a certain fixed point or *pole*, o, and is a function of the position-angles $(\theta, \phi, \theta', \phi')$ of these points with regard to the pole. If either point, as p, is taken as pole, the Laplacian (being always simply a function of $\cos\psi$) will reduce to a function of μ' alone, and its value is then obtained by taking $\mu = 1$ and $\phi - \phi' = 0$ in its *general* expression. In this case—i. e. when one of the two related points in the Laplacian is the pole—the Laplacian is called a *Legendre's coefficient*, which therefore expresses exactly the same thing as the Laplacian, but by a transformation of co-ordinates. In this special form these functions were employed by Legendre before Laplace used them in the general form.

350.] **Spherical Harmonics.** *To determine a homogeneous function of x, y, z, of the most general form, which satisfies the equation $\nabla^2 V = 0$.*

Firstly, such a function involves $2i+1$ arbitrary constants, because it contains $\frac{1}{2}(i+1)(i+2)$ terms; and $\nabla^2 V$, being a rational integral function of degree $i-2$, will contain $\frac{1}{2}i(i-1)$ separate terms. The condition that $\nabla^2 V$ should vanish for all values of x, y, z is that the coefficient of each of these $\frac{1}{2}i(i-1)$ is zero; so that we have $\frac{1}{2}i(i-1)$ equations between the $\frac{1}{2}(i+1)(i+2)$ coefficients. This leaves $2i+1$ of them independent.

Changing from Cartesian to polar co-ordinates, such a function will be of the form $r^i Y_i$, where Y_i is, of course, a rational, integral, and homogeneous function of μ, $\sqrt{1-\mu^2}\cos\phi$, and $\sqrt{1-\mu^2}\sin\phi$, and Y_i will satisfy the equation (ϵ), p. 238, or

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dY_i}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2 Y_i}{d\phi^2} + i(i+1)Y_i = 0. \qquad (1)$$

We can now show that a value of Y_i which is the product of a function of μ only and a function of ϕ only can be found to satisfy this equation.* Let $Y_i = M\Phi$, where M is a function of μ only and Φ a function of ϕ only. Then we have

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dM}{d\mu}\right\} + i(i+1)M + \frac{M}{1-\mu^2}\cdot\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0. \qquad (2)$$

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* This method is found in Ferrers's *Spherical Harmonics*, p. 78, a work which ought to be studied by the student who desires to pursue this subject further.

Assume
$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -n^2, \tag{3}$$
where n is a constant. Then
$$\Phi = A\cos n\phi + B\sin n\phi. \tag{4}$$

Equation (2) for M now becomes, putting k for $i(i+1)$,
$$(1-\mu^2)\frac{d^2M}{d\mu^2} - 2\mu\frac{dM}{d\mu} + \left(k - \frac{n^2}{1-\mu^2}\right)M = 0. \tag{5}$$

Now if $v \equiv \frac{d^i(\mu^2-1)^i}{d\mu^i}$, we have shown (last Art.) that
$$(1-\mu^2)\frac{d^2v}{d\mu^2} - 2\mu\frac{dv}{d\mu} + kv = 0\,; \tag{6}$$
and we proceed to show that χ can be determined so that the value $M = \chi\frac{d^n v}{d\mu^n}$ will satisfy the equation (5).

For brevity denote $\frac{d^n v}{d\mu^n}$ by v_n, $\frac{d^{n+1}v}{d\mu^{n+1}}$ by v_{n+1}, $\frac{d\chi}{d\mu}$ by χ', &c.

Then (5) becomes
$$(1-\mu^2)\chi v_{n+2} + 2\{(1-\mu^2)\chi' - \mu\chi\}v_{n+1}$$
$$+\left\{(1-\mu^2)\chi'' - 2\mu\chi' + \left(k - \frac{n^2}{1-\mu^2}\right)\chi\right\}v_n = 0. \tag{7}$$

Differentiate (6) n times, employing the theorem of Leibnitz. Then
$$(1-\mu^2)v_{n+2} - 2(1+n)\mu v_{n+1} + \{k - n(n+1)\}v_n = 0. \tag{8}$$

Now identifying (7) and (8), if possible, we have
$$(1-\mu^2)\chi' + n\mu\chi = 0, \tag{9}$$
$$(1-\mu^2)\chi'' - 2\mu\chi' + n\left\{n + 1 - \frac{n}{1-\mu^2}\right\}\chi = 0\,; \tag{10}$$
and since (10) is deducible from (9) by differentiation, the identification of (7) and (8) is possible. From (9) we have
$$\chi = a(1-\mu^2)^{\frac{n}{2}},$$
where a is any constant. Hence
$$M = a(1-\mu^2)^{\frac{n}{2}}\frac{d^{i+n}(\mu^2-1)^i}{d\mu^{i+n}}, \tag{α}$$
and the function
$$(1-\mu^2)^{\frac{n}{2}}\frac{d^{i+n}(\mu^2-1)^i}{d\mu^{i+n}}(A\cos n\phi + B\sin n\phi), \tag{β}$$

where n may obviously be *any integer from* 0 *to* i, satisfies the equation (1); and this function when multiplied by r^i is the type of rational integral functions of x, y, z satisfying Laplace's equation $\nabla^2 V = 0$.

All such functions are called *Spherical Harmonics*.

The coefficients of the various powers of r'/r in the expansion of $1/PP'$, which we have spoken of as *Laplacians*, are, of course, Spherical Harmonics particularized.

The function $r^i Y_i$ (which is a homogeneous function of x, y, z) is called a Solid Spherical Harmonic, or simply a Solid Harmonic, of the i^{th} degree; while the portion Y_i, which is a function of μ and ϕ, is called a Surface Harmonic.

Again (see Art. 329) corresponding to a Solid Spherical Harmonic $r^i Y_i$ of *positive* degree, i, there is a Solid Spherical Harmonic, Y/r^{i+1}, of negative degree, $-(i+1)$.

Any expression of the form (β) is called a *Tesseral Surface Harmonic* of degree i and order n.

When $n = 0$, the Tesseral Harmonic becomes

$$\frac{d^i(\mu^2-1)^i}{d\mu^i},$$

multiplied by a factor independent of μ, and this is called a *Zonal Harmonic* of the i^{th} degree. The Zonal Harmonic of the i^{th} degree becomes identical in form with the Laplacian when p' is taken as pole (Fig. 289), and in order that it may assume the value unity when $\mu = 1$, we take

$$P_i = \frac{1}{2^i \underline{|i}} \frac{d^i(\mu^2-1)^i}{d\mu^i}, \qquad (\gamma)$$

(last Art.) where we use P_i to denote the Zonal Harmonic of the i^{th} degree.

It is evident that the sum of all such terms as (β), each multiplied by an arbitrary constant, n, receiving all values from 0 to i both inclusive, will satisfy (1); and that this sum of terms gives us a function, Y_i, involving $2i+1$ arbitrary constants. It is, therefore, the function sought.

It is thus seen that the Zonal Harmonic P_i is the base or *source* of the general Spherical Harmonic of the i^{th} degree. Thus, for example, the Spherical Harmonic of the 3rd degree is derived from the source $\dfrac{d^3(\mu^2-1)^3}{d\mu^3}$,

i.e. from $120\mu^3 - 72\mu$, or, neglecting a numerical factor, from $5\mu^3 - 3\mu$; and this Harmonic will be the sum of the terms obtained by giving n the values 0, 1, 2, 3 in the expression

$$(1-\mu^2)^{\frac{n}{2}} \frac{d^n}{d\mu^n}(5\mu^3 - 3\mu) \,.\, (A\cos n\phi + B\sin n\phi).$$

It is therefore of the general form

$$A_0(5\mu^3 - 3\mu) + (1-\mu^2)^{\frac{1}{2}}(5\mu^2 - 1)(A_1\cos\phi + B_1\sin\phi)$$
$$+ (1-\mu^2)\mu(A_2\cos 2\phi + B_2\sin 2\phi)$$
$$+ (1-\mu^2)^{\frac{3}{2}}(A_3\cos 3\phi + B_3\sin 3\phi),$$

the coefficients $A_0, A_1, \ldots$ being all arbitrary constants.

The corresponding Solid Harmonic is obtained by multiplying this by r^3.

The homogeneity of the expression for Y_i as a function of μ, $\sqrt{1-\mu^2}\cos\phi$, $\sqrt{1-\mu^2}\sin\phi$ may not be at once apparent. For example, the term $A_0(5\mu^3 - 3\mu)$ comes from the function

$$A_0\{5z^3 - 3z(x^2+y^2+z^2)\}, \text{ or } A_0(2z^2 - 3x^2 - 3y^2)z,$$

from which the term in ϕ disappears in consequence of the relation $\sin^2\phi + \cos^2\phi = 1$.

We are now in a position to express the Laplacian L_i. Since it is a spherical surface harmonic of the i^{th} degree both in the co-ordinates (μ, ϕ) and in the co-ordinates (μ', ϕ') and involves both in identically the same way, its general term must be

$$A_n(1-\mu^2)^{\frac{n}{2}}(1-\mu'^2)^{\frac{n}{2}} \frac{d^n P_i}{d\mu^n} \frac{d^n P_i'}{d\mu'^n} \cos n(\phi - \phi'), \tag{δ}$$

where A_n is a factor independent of μ, μ', ϕ, ϕ'; and L_i is the sum of all such terms obtained by giving n values from 0 to i, inclusive.

For the purpose of actual calculation, it will be better to write the coefficient of $\cos n(\phi - \phi')$ in the form

$$C_n(1-\mu^2)^{\frac{n}{2}}(1-\mu'^2)^{\frac{n}{2}} \frac{d^{i+n}(\mu^2-1)^i}{d\mu^{i+n}} \frac{d^{i+n}(\mu'^2-1)^i}{d\mu'^{i+n}}. \tag{ϵ}$$

Since the determination of C_n is merely the analytical process of identifying the expression (ϵ) with the coefficient of $\cos n(\phi - \phi')$ in the value of L_i given in (ζ), Art. 349, we may obviously suppose $\mu = \mu'$. In this case (ϵ) becomes

$$C_n(1-\mu^2)^n \left[\frac{d^{i+n}(\mu^2-1)^i}{d\mu^{i+n}}\right]^2; \tag{11}$$

and the highest term in μ in this expression is

$$C_n(-1)^n(2i\,.\,2i-1\,\ldots\,i-n+1)^2\mu^{2i}. \tag{12}$$

Now using ω for $\phi-\phi'$, in this case

$$\lambda = \mu^2+(1-\mu^2)\cos\omega\,;$$

$$\therefore\ \lambda^2-1 = (2\sin\tfrac{1}{2}\omega)^2(\mu^2-1)(\mu^2\sin^2\tfrac{1}{2}\omega+\cos^2\tfrac{1}{2}\omega)$$
$$= (2\sin\tfrac{1}{2}\omega)^2(\xi^2\sin^2\tfrac{1}{2}\omega+\xi),$$

if we put $\mu^2-1=\xi$.

Again, $\dfrac{d}{d\lambda} = \dfrac{1}{2\sin^2\frac{1}{2}\omega}\cdot\dfrac{d}{d\xi}\cdot$ Hence the value of L_i becomes

$$\frac{1}{\lfloor i}\frac{d^i}{d\xi^i}(\xi^2\sin^2\tfrac{1}{2}\omega+\xi)^i. \tag{13}$$

We shall determine C_n by equating the coefficient of the highest power of μ (or of ξ) in the coefficient of $\cos n\omega$ in (13) to the expression (12). Now obviously the highest term in ξ in (13) is

$$\frac{2i\,.\,2i-1\,\ldots\,i+1}{\lfloor i}\sin^{2i}\frac{\omega}{2}\cdot\xi^i,$$

so that the highest term in μ is

$$\frac{2i\,.\,2i-1\,\ldots\,i+1}{\lfloor i}\sin^{2i}\frac{\omega}{2}\cdot\mu^{2i}\,; \tag{14}$$

and the coefficient of $\cos n\omega$ in this must be identical with (12).

But, by elementary Trigonometry,

$$(-1)^i\,.\,2^{2i}\sin^{2i}\theta = 2\cos 2i\theta+\ldots$$
$$+(-1)^p\,2\,\frac{2i\,.\,2i-1\,\ldots\,2i-p+1}{\lfloor p}\cos(2i-2p)\theta+\ldots. \tag{15}$$

Hence if $p=i-n$, we have

$$(-1)^i\,.\,2^{2i}\sin^{2i}\frac{\omega}{2} = 2\cos i\omega+\ldots$$
$$+(-1)^{i-n}\,2\,\frac{2i\,.\,2i-1\,\ldots\,i+n+1}{\lfloor i-n}\cos n\omega+\ldots\,;$$

therefore by (14),

$$C_n(2i\,.\,2i-1\,\ldots\,i-n+1)^2$$
$$= \frac{1}{2^{2i-1}}\cdot\frac{2i\,.\,2i-1\,\ldots\,i+1}{\lfloor i}\cdot\frac{2i\,.\,2i-1\,\ldots\,i+n+1}{\lfloor i-n},$$

or

$$C_n = \frac{2}{(2^i\lfloor i)^2}\frac{\lfloor i-n}{\lfloor i+n}. \tag{η}$$

As before said, n is to receive all values from 0 to i, and when $n = i$, the expression $\underline{|i-n}$ is to be taken as unity.

When $n = 0$, the value of C given by (η) must be halved, because there is a middle term in (15), which is independent of ω, and it is not multiplied by the 2 which affects all the other terms.

Hence for the Laplacian of the i^{th} order, we have

$$L_i = \frac{2}{(2^i \underline{|i})^2} \sum_{n=0}^{n=i} \frac{\underline{|i-n}}{\underline{|i+n}} (1-\mu^2)^{\frac{n}{2}} (1-\mu'^2)^{\frac{n}{2}} \frac{d^{i+n}(\mu^2-1)^i}{d\mu^{i+n}} \frac{d^{i+n}(\mu'^2-1)^i}{d\mu'^{i+n}} \cos n(\phi-\phi'), \ldots \quad (\theta)$$

in which the first term (corresponding to $n = 0$) must be halved.

351.] **Fundamental Property of Spherical Harmonics.** If Y_i and $Z_{i'}$ are any two Spherical Harmonics of degrees i and i',

$$\int_{-1}^{1} \int_{0}^{2\pi} Y_i Z_{i'} \, d\mu \, d\phi = 0, \quad (\alpha)$$

or, in other words, $\int Y_i Z_{i'} \, dS$ extended over a sphere of unit radius is zero—that is, *the spherical surface-integral of the product of any two Spherical Harmonics of different degrees is zero.*

For, $\nabla^2 (r^i Y_i) = 0$, which gives (1) Art. 350; and (Art. 329)

$$\nabla^2 Y_i = \frac{1}{r^2}\left[\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dY_i}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2 Y_i}{d\phi^2}\right] = -i(i+1)\frac{Y_i}{r^2}.$$

Similarly $\nabla^2 Z_{i'} = -i'(i'+1)\frac{Z_{i'}}{r^2}$. Now in Green's Equation, (β), Art. 346, let $V = Y_i$, $U = Z_{i'}$, and let the integrations be extended through the volume and over the surface of a sphere of radius a. Then, the centre of this sphere being the origin of the co-ordinates (r, μ, ϕ), it is clear that $\frac{dY_i}{dn} = 0 = \frac{dZ_{i'}}{dn}$. Hence we have

$$[i(i+1) - i'(i'+1)] \int Y_i Z_{i'} \frac{d\Omega}{r^2} = 0.$$

But $d\Omega = r^2 dr \, d\mu \, d\phi$, and $Y_i Z_{i'}$ does not involve r; therefore we have

$$[i(i+1) - i'(i'+1)] \, a \iint Y_i Z_{i'} \, d\mu \, d\phi = 0,$$

which gives the result (α) *except when* $i = i'$.

We postpone for a moment the investigation of the value of the double integral when i and i' are the same.

352.] **Spherical Harmonic Expansion of a Function of μ and ϕ.** Let P (Fig. 277, p. 152) be any point outside a spherical surface of radius a, at a distance R from the centre, and let r be the distance, PQ, between P and any point on the surface. Then if dS is an element of surface at Q, we have

$$\int \frac{dS}{r^n} = \frac{2\pi a}{(n-2)R}\left\{\frac{1}{(R-a)^{n-2}} - \frac{1}{(R+a)^{n-2}}\right\},$$

as is easily found by using for dS the expression (A), p. 152.

Hence wherever P is, we have

$$(R^2-a^2)^{n-2}\int\frac{dS}{r^n} = \frac{2\pi a}{(n-2)R}\{(R+a)^{n-2}-(R-a)^{n-2}\}. \qquad (\alpha)$$

If P is at A, i.e. on the surface, its distance from one of the surface elements becomes zero, and $R = a$; so that the left-hand side of (α) assumes an apparently indeterminate form. (See, however, the remarks, p. 149.) But it is really finite and, as the right-hand side shows, equal to $\frac{2^{n-1}\pi}{n-2}a^{n-2}$.

Of course, of the whole surface of the sphere it is only an infinitely small element of the tangent plane at A that contributes to the integral, each element, $(R^2-a^2)\frac{dS}{r^n}$, of the integral being zero when r is appreciable, R being equal to a.

Hence
$$\left[(R^2-a^2)^{n-2}\int\frac{dS}{r^n}\right]_{R=a} = \frac{2^{n-1}\pi}{n-2}a^{n-2}. \qquad (\beta)$$

Again, if U is any function of the co-ordinates of a point, (x, y, z) or (r, μ, ϕ), the value of $(R^2-a^2)^{n-2}\int\frac{UdS}{r^n}$ when P is at A is obviously $U_A \times \left[(R^2-a^2)^{n-2}\int\frac{dS}{r^n}\right]_{R=a}$, assuming that when r is anything different from zero, U is never $= \infty$; because in this case it is only an infinitely small element of the tangent plane at A that contributes to the integral. In other words, if for no point, Q, on the sphere U is ∞, we have

$$\left[(R^2-a^2)^{n-2}\int\frac{UdS}{r^n}\right]_{R=a} = \frac{2^{n-1}\pi}{n-2}a^{n-2}\,U_A, \qquad (\gamma)$$

where U_A denotes the value of U at A.

We may assume U to be, definitely, a function of μ and ϕ, which does not become infinite at any point. Take the case $n = 3$. Then denoting by dashes the values of functions at variable points on the sphere, such as p' (Fig. 289, p. 239), the functions without dashes belonging to a fixed point, p, on the sphere, we have

$$\left[(R^2-a^2)\int\frac{U'dS}{r^3}\right]_{R=a} = 4\pi a U. \tag{δ}$$

Now taking $r^2 = R^2 - 2aR\lambda + a^2$, where $\lambda \equiv \cos\psi$ (Fig. 289), we have $\dfrac{dr}{dR} = \dfrac{R-a\lambda}{r} = \dfrac{R^2-a^2+r^2}{2Rr}$; and

$$\frac{d\frac{1}{r}}{dR} = -\frac{1}{r^2}\frac{dr}{dR} = -\frac{R^2-a^2}{2Rr^3} - \frac{1}{2Rr};$$

therefore

$$\frac{R^2-a^2}{r^3} = -2R\frac{d\frac{1}{r}}{dR} - \frac{1}{r}. \tag{1}$$

Now $$\frac{1}{r} = \frac{1}{R}\left(L_0 + L_1\frac{a}{R} + L_2\frac{a^2}{R^2} + \ldots + L_i\frac{a^i}{R^i} + \ldots\right),$$

therefore (1) becomes

$$\frac{R^2-a^2}{r^3} = \frac{1}{R}\left[L_0 + 3L_1\frac{a}{R} + 5L_2\frac{a^2}{R^2} + \ldots + (2i+1)L_i\frac{a^i}{R^i} + \ldots\right]. \tag{2}$$

Multiplying both sides of this equation by $U'dS$, that is by $U'a^2d\mu'd\phi'$, we have, whatever be the value of R,

$$(R^2-a^2)\int\frac{U'dS}{r^3} = \frac{a^2}{R}\left[\iint L_0U'd\mu'd\phi' + 3\frac{a}{R}\iint L_1U'd\mu'd\phi' + \ldots \right.$$
$$\left. + (2i+1)\frac{a^i}{R^i}\iint L_iU'd\mu'd\phi' + \ldots\right]; \tag{ϵ}$$

the limits of μ being 1 and -1, and those of ϕ being 0 and 2π.

Now put $R = a$ in this equation, and we have, by (δ),

$$\iint L_0U'd\mu'd\phi' + 3\iint L_1U'd\mu'd\phi' + \ldots$$
$$+ (2i+1)\iint L_iU'd\mu'd\phi' + \ldots = 4\pi U. \tag{ζ}$$

As a particular case let $U = Y_i$ = any Spherical Harmonic of the i^{th} degree. Then, since L_i is also a Spherical Harmonic of the same degree, every term except one in (ζ) vanishes by last Article, and we have

$$\int_{-1}^{1}\int_0^{2\pi} L_iY_i'd\mu'd\phi' = \frac{4\pi}{2i+1}Y_i. \tag{η}$$

which expresses a most remarkable property of a Laplacian, namely—*If over a sphere there be taken the surface-integral of the product of any Spherical Harmonic and the Laplacian of the same degree, i, with reference to any fixed point on the sphere, the result is the value of the given Spherical Harmonic at this fixed point, multiplied by* $4\pi/(2i+1)$.

This result enables us to express any function of μ and ϕ which does not become infinite for any values of μ and ϕ in the form of a series of Spherical Harmonics.

Thus, let U be the given function, which belongs to the fixed point p, Fig. 289, and let

$$U = Y_0 + Y_1 + Y_2 + \ldots + Y_i + \ldots, \tag{3}$$

the quantities Y_0, Y_1, ... to be determined.

To determine Y_i, substitute running co-ordinates μ', ϕ' (those of p') in both sides of (3), multiply by L_i, where L_i is the Laplacian of the i^{th} degree for p and p', and integrate.

Then by (η) and last Article, we have

$$Y_i = \frac{2i+1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} L_i U' d\mu' d\phi', \tag{4}$$

and by giving i all values from 0 upwards, we find the series of Y's.

The above method is due to Ivory (see Todhunter's *History of the Theories of Attraction*, &c., vol. ii, p. 261).

It is scarcely necessary to observe that a given function of μ and ϕ can be expanded in only one way in a series of Spherical Harmonics; for every Harmonic of the series is perfectly and uniquely determined by (4).

There is, however, another method by which a function of μ and ϕ can be expanded in a series of Surface Harmonics without integration and the employment of Laplacians. To explain this method, suppose $F(x, y, z)$ to be any rational, integral, and homogeneous function of x, y, z of the n^{th} degree. Then this function can be expressed in the form

$$F(x, y, z) = S_n + r^2 S_{n-2} + r^4 S_{n-4} + \ldots, \tag{5}$$

where S_n, S_{n-2}, ... are Solid Harmonics of degrees n, $n-2$, ..., the last term being $r^n S_0$ if n is even, and $r^{n-1} S_1$ if n is odd.

Terms involving odd powers of r cannot appear in (5); for we can easily prove that

$$\nabla^2 . r^p S_q = p(p+2q+1) r^{p-2} S_q, \tag{6}$$

S_q being a Solid Harmonic of degree q. Now if a term rS_{n-1} occurred in (5) and we performed the operation ∇^2 on both sides, this term would give rise to the only term in $1/r$ in the equation. Hence this term must be absent. Similarly a term $r^3 S_{n-3}$ could not occur; for, after performing ∇^2 twice on each side of (5), we should have the same result as before. Successive performances of the operation ∇^2 on (5) will give the required Harmonics S_0, S_2, S_4, ... if n is even, or S_1, S_3, S_5, ... if n is odd, in this order.

For example, to express xyz^3 in the form (5). Let

$$xyz^3 = S_5 + r^2 S_3 + r^4 S_1, \tag{7}$$

$$\therefore\quad 6xyz = 18 S_3 + 28 r^2 S_1,$$

$$0 = S_1,$$

by performing ∇^2 twice. Hence $S_3 = \frac{1}{3}xyz$, and (7) gives

$$S_5 = \tfrac{1}{3}xyz\,(2z^2 - x^2 - y^2).$$

Now this enables us to exhibit $\sin^2\theta\cos^3\theta\sin\phi\cos\phi$ as a series of Surface Harmonics; for when this is multiplied by r^5, it becomes xyz^3, and we have

$$r^5 \sin^2\theta\cos^3\theta\sin\phi\cos\phi = \tfrac{1}{3}xyz\,(2z^2 - x^2 - y^2) + \tfrac{1}{3}r^2 xyz,$$

so that the given expression in θ and ϕ is of the form $Y_5 + Y_3$,

where $\quad Y_5 = \dfrac{1}{3r^5}xyz\,(2z^2 - x^2 - y^2)$, and $Y_3 = \dfrac{1}{3r^3}xyz$,

$$\therefore\quad Y_5 = \mu\,(1-\mu^2)\,(\mu^2 - \tfrac{1}{3})\sin\phi\cos\phi,$$

and

$$Y_3 = \tfrac{1}{3}\mu\,(1-\mu^2)\sin\phi\cos\phi.$$

353.] **Value of** $\displaystyle\int_{-1}^{1}\int_{0}^{2\pi} Y_i Z_i \,d\mu\,d\phi$. The spherical surface-integral of the product of two Spherical Harmonics of the *same* degree is found by Laplace very simply from the results of last Article.

Denoting by M_n the factor in μ in (β), p. 242, we may write

$$Y_i = A_0 M_0 + M_1 (A_1\cos\phi + B_1\sin\phi) + \dots$$
$$+ M_n (A_n \cos n\phi + B_n \sin n\phi) + \dots, \tag{1}$$

$$Z_i = a_0 M_0 + M_1 (a_1\cos\phi + b_1\sin\phi) + \dots$$
$$+ M_n (a_n \cos n\phi + b_n \sin n\phi) + \dots; \tag{2}$$

the two functions differing simply in their constants A's, B's, a's, b's.

Now since in the integration ϕ runs from 0 to 2π, it is obvious that the integrals of all products will vanish except those of the type

$$M_n^2 (A_n \cos n\phi + B_n \sin n\phi)(a_n \cos n\phi + b_n \sin n\phi),$$

and the integral of this is

$$\pi (A_n a_n + B_n b_n) \,.\, M_n^2, \tag{α}$$

but for the first term the integral will be

$$2\pi A_0 a_0 M_0^2. \tag{α'}$$

We have therefore to find $\int_{-1}^{1} M_n^2 d\mu$, which Laplace finds as follows. With the notation of Art. 350, write

$$L_i = C_0 M_0 M_0' + C_1 M_1 M_1' \cos(\phi - \phi') + \dots$$
$$C_n M_n M_n' \cos n(\phi - \phi') + \dots. \tag{3}$$

Put running co-ordinates into (1), multiply by (3) and take the surface-integral over a sphere. Then we have simply a sum of terms of the type

$$C_n M_n \int_{-1}^{1} \int_{0}^{2\pi} M_n'^2 (A_n \cos n\phi' + B_n \sin n\phi') \cos n(\phi - \phi') \, d\mu' \, d\phi'.$$

Performing the integration in ϕ', this becomes

$$\pi C_n M_n (A_n \cos n\phi + B_n \sin n\phi) \,.\, \int_{-1}^{1} M_n'^2 d\mu'. \tag{β}$$

The sum of all terms of the type (β) is therefore the value of $\iint L_i Y_i' d\mu' d\phi'$. But (last Article) this $= 4\pi Y_i/(2i+1)$; therefore, by identification of coefficients of like terms,

$$C_n \int_{-1}^{1} M_n'^2 d\mu' = \frac{4}{2i+1}.$$

Putting for C_n its value given in (η), p. 245,

$$\int_{-1}^{1} M_n^2 d\mu = \frac{2(2^i \lfloor i)^2}{2i+1} \cdot \frac{\lfloor i+n}{\lfloor i-n}, \tag{γ}$$

which holds, without change, for the case $n = 0$, notwithstanding that the value of C_n (Art. 350) must be halved when $n = 0$; because in the product of (1) and (3) the term independent of ϕ' is $C_0 A_0 M_0 M_0'^2$, which in the integration will give

$$2\pi C_0 A_0 M_0 \int_{-1}^{1} M_0'^2 d\mu'.$$

Hence we have

$$\int_{-1}^{1}\int_{0}^{2\pi} Y_i Z_i d\mu d\phi = \frac{2^{2i+1}}{2i+1}\pi(|\underline{i})^2 \sum_0^i \frac{|\underline{i+n}}{|\underline{i-n}}(A_n a_n + B_n b_n), \quad (\delta)$$

the first term (that corresponding to $n = 0$) being doubled, by (α').

Putting $a_n = A_n$, $b_n = B_n$, we obtain the value of

$$\int_{-1}^{1}\int_{0}^{2\pi} Y_i^2 d\mu d\phi.$$

354.] **Table of Laplacians.** For convenience of reference the following table of the Laplacians as far as L_4 is given; but, to save space, we give in the coefficients of

$$\cos(\phi-\phi'), \cos 2(\phi-\phi'), \ldots$$

only the portion which depends on μ. This portion is to be multiplied by exactly the same function of μ'. Thus, for example, in L_3 the coefficient of $\cos 2(\phi-\phi')$ is $\frac{15}{4}\mu(1-\mu^2)\,.\,\mu'(1-\mu'^2)$, of which only the part $\frac{15}{4}\mu(1-\mu^2)$ is given in the column under $\cos 2(\phi-\phi')$; in L_4 the term involving $\cos 3(\phi-\phi')$ is

$$\tfrac{35}{8}\mu(1-\mu^2)^{\frac{3}{2}}\,.\,\mu'(1-\mu'^2)^{\frac{3}{2}}\cos 3(\phi-\phi')\,;\ \&c.$$

Values of i.	Term in μ only.	Coefficient of $\cos(\phi-\phi')$	Coefficient of $\cos 2(\phi-\phi')$	Coefficient of $\cos 3(\phi-\phi')$	Coefficient of $\cos 4(\phi-\phi')$
0	1				
1	μ	$(1-\mu^2)^{\frac{1}{2}}$			
2	$\frac{1}{4}(3\mu^2-1)$	$3\mu(1-\mu^2)^{\frac{1}{2}}$	$\frac{3}{4}(1-\mu^2)$		
3	$\frac{1}{4}(5\mu^3-3\mu)$	$\frac{3}{8}(1-\mu^2)^{\frac{1}{2}}(5\mu^2-1)$	$\frac{15}{4}\mu(1-\mu^2)$	$\frac{5}{8}(1-\mu^2)^{\frac{3}{2}}$	
4	$\frac{1}{64}(35\mu^4-30\mu^2+3)$	$\frac{5}{8}(1-\mu^2)^{\frac{1}{2}}(7\mu^3-3\mu)$	$\frac{5}{16}(1-\mu^2)(7\mu^2-1)$	$\frac{35}{8}\mu(1-\mu^2)^{\frac{3}{2}}$	$\frac{35}{64}(1-\mu^2)^2$

The column of terms in μ only gives the values of the first five Legendre's Coefficients, with the numerical coefficient squared; thus

$$\begin{aligned} P_0 &= 1, \\ P_1 &= \mu, \\ P_2 &= \tfrac{1}{2}(3\mu^2-1), \\ P_3 &= \tfrac{1}{2}(5\mu^3-3\mu), \\ P_4 &= \tfrac{1}{8}(35\mu^4-30\mu^2+3). \end{aligned}$$

The Solid Zonal Harmonics play exactly the same part with regard to the Potential of a body symmetrical about an axis (see Example 2 following) as the variables x, y, z do with respect to the equation of a plane surface, the equation of such a surface consisting of the sum of these co-ordinates each multiplied

by a constant, and these constants depending on the position of the plane. Similarly, the Potential of a symmetrical body at any point consists of the sum of a number of these Harmonics each multiplied by a coefficient which depends on the shape and law of density of the body and not on the position of the attracted particle. In fact, Zonal Harmonics may be considered as the *running co-ordinates of the Potential* of such a body.

Of the Zonal Surface Harmonics P_1, P_3, P_5, ... are all of the form $\mu f(\mu^2)$, and P_2, P_4, P_6, ... are all of the form $f(\mu^2)$. For, in the identity

$$(1-2\mu x+x^2)^{-\frac{1}{2}} = P_0+P_1 x+P_2 x^2+\ldots+P_i x^i+\ldots \qquad (\alpha)$$

change μ to $-\mu$, and we get

$$(1+2\mu x+x^2)^{-\frac{1}{2}} = P_0+P_1' x+P_2' x^2+\ldots+P_i' x^i \ldots, \qquad (\beta)$$

where P_1', P_2', ... denote the values of the Harmonics when μ is changed to $-\mu$. Again, changing only the sign of x,

$$(1+2\mu x+x^2)^{-\frac{1}{2}} = P_0-P_1 x+P_2 x^2-P_3 x^3+\ldots. \qquad (\gamma)$$

Identifying the results (β) and (γ), we see that P_1, P_3, ... all change sign with μ; while P_2, P_4, ... do not. Therefore, &c.

EXAMPLES.

1. Find $\int_{-1}^{1}\int_{0}^{2\pi} L_i^2 d\mu' d\phi'$.

Let P be any point outside a sphere of radius a, at a distance R from the centre, and Q any point on the surface; find $\int dS/r^2$ over the surface, where $r = PQ$.

Now $\qquad 1/r = 1/R\,(L_0+L_1 a/R+\ldots+L_i a^i/R^i+\ldots)$,

and $1/r^2$ will involve such terms as $L_m L_i$ which will vanish (Art. 351) in the integration. Hence, obviously, since $dS = -a^2 d\mu' d\phi'$, we have

$$\int \frac{dS}{r^2} = \ldots \frac{a^{2i+2}}{R^{2i+2}} \iint L_i^2 d\mu' d\phi' + \ldots.$$

The integrations on both sides are taken over the whole surface of the sphere, and therefore that on the left may be taken in strips about P, while that on the right may be taken in strips round the pole from which μ, μ', ϕ, ϕ' are measured.

But $dS = 2\pi a r dr/R$ (see p. 152); therefore the left-hand side is $2\pi \frac{a}{R} \log \frac{R+a}{R-a}$. Develop this in a series ascending by powers of $\frac{a}{R}$, and equate the coefficients of $\left(\frac{a}{R}\right)^{2i+2}$ on both sides (since the develop-

ment holds for all values of R), and we have

$$\int_{-1}^{1}\int_{0}^{2\pi} L_i^2\, d\mu'\, d\phi' = \frac{4\pi}{2i+1}.$$

The result is therefore quite independent of the pole o (Fig. 289) from which μ and μ' are measured, and is the same as if the line OP (or Op) is the axis of θ, or p the pole of the Laplacian.

2. Prove the theorem of Legendre (Art. 348) by Spherical Harmonics.

Taking the centre of the solid as origin, and axis of revolution as that from which θ is measured, let (R, μ, ϕ) be the co-ordinates of the attracted particle, P, (r', μ', ϕ') those of any point, P', inside the solid, ρ the density of the solid at P', and γ the constant of gravitation. Then V, the Potential at P, is given by the equation

$$V = \gamma \iiint \frac{\rho r'^2 dr'\, d\mu'\, d\phi'}{PP'}.$$

Now, assuming the distance of P from the centre to be greater than that of every point P' in the solid, $1/PP'$ may be developed in the convergent series (β), p. 238. Hence

$$V = \gamma \iiint \left(\frac{L_0}{R} + L_1 \frac{r'}{R^2} + \dots + L_i \frac{r'^i}{R^{i+1}} + \dots\right)\rho r'^2 dr'\, d\mu'\, d\phi'. \quad (1)$$

But by hypothesis ρ is a function of r' and μ' only; and if when r' is produced out to meet the surface of the solid its value is R', this latter will be simply a function of μ', and will not involve ϕ'. Take the general term of the series (1), and first perform the integration in r' from 0 to R', taking the term

$$\int_0^{R'} \rho r'^{i+2} dr' = \chi(\mu'),$$

where the form of χ is unknown if the shape of the surface and the law of density are not given. Then we have

$$V = \dots \frac{\gamma}{R^{i+1}} \int_{-1}^{1}\int_{0}^{2\pi} L_i \chi(\mu') d\mu'\, d\phi' + \dots. \quad (2)$$

Now perform the integration in ϕ'. We shall have simply

$$\int_0^{2\pi} L_i\, d\phi',$$

which, of course, reduces to the first term of L_i, and is therefore (Art. 350) $\dfrac{2\pi}{(2^i \lfloor i)^2} \dfrac{d^i(\mu^2-1)^i}{d\mu^i} \dfrac{d^i(\mu'^2-1)^i}{d\mu'^i}$. Hence

$$V = \dots \frac{2\pi}{(2^i \lfloor i)^2} \frac{d^i(\mu^2-1)^i}{d\mu^i} \frac{\gamma}{R^{i+1}} \int_{-1}^{1} \frac{d^i(\mu'^2-1)^i}{d\mu'^i} \chi(\mu') d\mu' + \dots. \quad (3)$$

Let v be the Potential at a point on the axis distant z from the centre, so that $R = z$, $\mu = 1$. Then (3) gives

$$v = \ldots \frac{2\pi}{(2^i \lfloor i)^2} \cdot 2^i \lfloor i \cdot \frac{\gamma}{z^{i+1}} \int_{-1}^{1} \frac{d^i (\mu'^2 - 1)^i}{d\mu'^i} \chi(\mu') d\mu' + \ldots. \qquad (4)$$

But supposing, as we do, that v is known for all points on the axis, let it be expanded from the given form in a series, so that

$$v = \frac{a_0}{z} + \frac{a_1}{z^2} + \ldots + \frac{a_i}{z^{i+1}} + \ldots. \qquad (5)$$

Then identifying (4) and (5), we have

$$\frac{2\pi}{(2^i \lfloor i)^2} \gamma \int_{-1}^{1} \frac{d^i (\mu'^2 - 1)^i}{d\mu'^i} \chi(\mu') d\mu' = \frac{a_i}{2^i \lfloor i},$$

so that the unknown coefficient in (3) is thus known. Hence

$$V = \gamma \frac{M}{R} + \ldots \frac{a^i}{2^i \lfloor i} \frac{d^i (\mu^2 - 1)^i}{d\mu^i} \cdot \frac{1}{R^{i+1}} + \ldots, \qquad (6)$$

the first term being easily seen to be $\frac{\gamma M}{R}$, where M is the mass of the solid. If $P_0, P_1, \ldots$ denote, as before, the several Zonal Harmonics, or Legendre's coefficients, for the attracted point with reference to the axis of the solid, we may write, by (γ), p. 243,

$$V = \frac{1}{R} \left\{ a_0 P_0 + \frac{a_1 P_1}{R} + \frac{a_2 P_2}{R^2} + \ldots + \frac{a_i P_i}{R^i} + \ldots \right\}. \qquad (7)$$

The components of attraction at P are of course known from this value of V.

Thus, then, the Potential of a solid symmetrical about an axis, both as regards shape and density, is in all cases given by a series of Solid Zonal Harmonics (of either positive or negative degrees, according as the point considered is internal or external), in which series the only things unknown are the coefficients $a_0, a_1, \ldots$ and the values of these depend on the nature of the particular attracting body.

3. Application of this method to the case of a uniform circular ring.

The Potential at a point distant z from the centre on the axis of the ring (that is, the line through its centre perpendicular to its plane) is given by the equation $v = 2\pi\gamma\rho k a / \sqrt{z^2 + a^2}$, where ρ, k, a are the density, area of transverse section, and radius of the ring. If M is the mass of the ring, $M = 2\pi\rho k a$; and if the point is at a distance $> a$ from the centre, we have

$$V = \gamma \frac{M}{z} \left\{ 1 - \frac{1}{2} \frac{a^2}{z^2} + \frac{1 \cdot 3}{2 \cdot 4} \frac{a^4}{z^4} - \ldots + (-1)^i \frac{1 \cdot 3 \cdot 5 \ldots 2i-1}{2 \cdot 4 \cdot 6 \ldots 2i} \frac{a^{2i}}{z^{2i}} + \ldots \right\}.$$

Hence, by last example, if the attracted particle is anywhere off the axis, at a distance r from the centre ($r > a$),

$$V = \gamma \frac{M}{r} \left\{ 1 - \tfrac{1}{2} P_2 \frac{a^2}{r^2} + \ldots + (-1)^i \frac{1.3\ldots 2i-1}{2.4\ldots 2i} P_{2i} \frac{a^{2i}}{r^{2i}} + \ldots \right\}.$$

If z is $< a$, the radical $(z^2 + a^2)^{-\frac{1}{2}}$ must be expanded in direct powers of z, and for a point anywhere at a distance $< a$,

$$V = \gamma \frac{M}{a} \left\{ 1 - \tfrac{1}{2} P_2 \frac{r^2}{a^2} + \ldots + (-1)^i \frac{1.3\ldots 2i-1}{2.4\ldots 2i} P_{2i} \frac{r^{2i}}{a^{2i}} + \ldots \right\}.$$

If the point is at the distance a from the centre, it is easy to prove that $V = \dfrac{\gamma M}{2\pi a \sqrt{2}} \displaystyle\int_0^{2\pi} \frac{d\phi}{\sqrt{1 - \sin\theta \cos\phi}}$, θ being the angle between the axis of the ring and the line joining the point to the centre. This is equivalent to the convergent series

$$V = \gamma \frac{M}{a\sqrt{2}} \left\{ 1 + \frac{1.3}{2.4} \cdot \tfrac{1}{2} \sin^2\theta + \frac{1.3.5.7}{2.4.6.8} \cdot \frac{1.3}{2.4} \sin^4\theta + \ldots \right.$$
$$\left. + \frac{1.3.5\ldots 4n-1}{2.4.6\ldots 4n} \cdot \frac{1.3.5\ldots 2n-1}{2.4.6\ldots 2n} \sin^{4n}\theta + \ldots \right\}.$$

Of course in this and in all similar examples, the value of V for a general position of the attracted particle, P, can be written down *in virtue of Legendre's Theorem solely* (Art. 348) by first calculating v, and in its expression replacing any such term as k/z^i by kP_{i-1}/r^i, because this latter satisfies the equation $\nabla^2 V = 0$, and it coincides with the former when P is on the axis, since $\mu = 1$, $P_1 = P_2 = \ldots = P_i = 1$. The expression thus obtained (somewhat tentatively) can, by Legendre's Theorem, be none other than the Potential sought.

4. Application to a uniform circular plate.

The position of the attracted particle being at a distance z from the centre on the axis of the plate, $v = 2\gamma \dfrac{M}{a^2}(\sqrt{z^2 + a^2} - z)$. When $z > a$, we have

$$v = 2\gamma \frac{M}{a^2} \left\{ \frac{1}{2} \cdot \frac{a^2}{z} - \frac{1}{2^2} \cdot \frac{1}{\underline{|2}} \frac{a^4}{z^3} + \ldots \right.$$
$$\left. + (-1)^{i-1} \frac{1}{2^i} \cdot \frac{1.3.5\ldots 2i-3}{\underline{|i}} \frac{a^{2i}}{z^{2i-1}} + \ldots \right\}.$$

Hence

$$V = 2\gamma \frac{M}{a^2} \left\{ \frac{1}{2} \frac{a^2}{r} - \frac{1}{2^2} \frac{1}{\underline{|2}} \frac{P_2 a^4}{r^3} + \ldots \right.$$
$$\left. + (-1)^{i-1} \frac{1}{2^i} \frac{1.3\ldots 2i-3}{\underline{|i}} \frac{P_{2i-2} a^{2i}}{r^{2i-1}} + \ldots \right\}.$$

When $z<a$, we easily find

$$V = 2\gamma\frac{M}{a^2}\left\{a - P_1 r + \tfrac{1}{2}P_2\frac{r^2}{a} + \dots \right.$$
$$\left. + (-1)^{i-1}\frac{1}{2^i}\frac{1\,.\,3\dots 2i-3}{\underline{|i}}P_{2i}\frac{r^{2i}}{a^{2i-1}} + \dots\right\}.$$

5. To find the conical angle subtended at any point, P, by a given circle.

Draw the axis of the circle, i.e. a perpendicular to its plane through its centre, O. Let $OP = r$, $a =$ radius of circle. Now if P were on the axis at a distance z from O, we should have

$$\omega_0 = 2\pi\left(1 - z/\sqrt{z^2+a^2}\right), \tag{1}$$

ω_0 being the conical angle subtended at the point; and, since conical angles satisfy all the equations of Potential functions, the theorem of Legendre applies to them.

Developing (1) in powers of a/z or z/a, according as z is $>$ or $<$ a, we have

$$\omega_0 = 2\pi\left\{\tfrac{1}{2}\frac{a^2}{z^2} - \frac{1\,.\,3}{2\,.\,4}\frac{a^4}{z^4} + \dots - (-1)^i\frac{1\,.\,3\dots 2i-1}{2\,.\,4\dots 2i}\frac{a^{2i}}{z^{2i}} + \dots\right\}, \tag{2}$$

$$\omega_0 = 2\pi\left\{1 - \frac{z}{a} + \tfrac{1}{2}\frac{z^3}{a^3} - \dots - (-1)^i\frac{1\,.\,3\dots 2i-1}{2\,.\,4\dots 2i}\frac{z^{2i+1}}{a^{2i+1}} + \dots\right\}. \tag{3}$$

Hence when P is off the axis we have in these two cases, respectively,

$$\omega = 2\pi\left\{\tfrac{1}{2}\frac{P_1 a^2}{r^2} - \frac{1\,.\,3}{2\,.\,4}\frac{P_3 a^4}{r^4} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{P_5 a^6}{r^6} - \dots\right\}, \tag{4}$$

$$\omega = 2\pi\left\{1 - \frac{P r}{a} + \tfrac{1}{2}\frac{P_3 r^3}{a^3} - \frac{1\,.\,3}{2\,.\,4}\frac{P_5 r^5}{a^5} + \dots\right\}. \tag{5}$$

6. Find the conical angle subtended at a point 10 feet distant from the centre of a circle 1 foot in radius, the colatitude of the point with reference to the axis of the circle being $\frac{1}{3}\pi$.

Ans. $\pi \times \cdot 0050328$, nearly.

7. If P_i is the Zonal Surface Harmonic of the i^{th} degree (Legendre's coefficient), show that

$$\mu\frac{dP_i}{d\mu} - iP_i = \frac{dP_{i-1}}{d\mu}. \tag{α}$$

We have by definition

$$1/\sqrt{1-2\mu x + x^2} = P_0 + P_1 x + \dots + P_{i-1}x^{i-1} + P_i x^i + \dots. \tag{1}$$

Denote the radical by T, and differentiate both sides with regard to x. Then

$$(\mu - x)/T^3 = P_1 + \dots + iP_i x^{i-1} + \dots. \tag{2}$$

Differentiate (1) with respect to μ; then

$$\frac{x}{T^3} = \dots x^{i-1}\frac{dP_{i-1}}{d\mu} + x^i\frac{dP_i}{d\mu} + \dots, \tag{3}$$

Multiplying (2) by x and (3) by $\mu - x$, we obtain two series which must be identical; and equating the coefficients of x^i in them, we have at once the result (α).

This result enables us to write down the values of the successive Zonal Harmonics when the first is known. For treating (α) as a linear differential equation for P_i, we have

$$P_i = \mu^i \left\{ C + \int \frac{1}{\mu^{i+1}} \frac{dP_{i-1}}{d\mu} \, d\mu \right\}. \qquad (4)$$

Then, as $P_0 = 1$, this gives $P_1 = C\mu$, and each P is to be unity when $\mu = 1$; therefore $C = 1$. Similarly P_2 is deduced from P_1; &c.

The expression (γ), p. 243, gives, however, the values of the Harmonics directly, and is the most convenient form for actual calculation.

8. Prove that $\quad (i+1)P_{i+1} - (2i+1)\mu P_i + iP_{i-1} = 0.$

Divide (1) by (2) and equate coefficients of like powers.

9. Prove that $$(1-\mu^2)\frac{dP_i}{d\mu} + i\mu P_i = iP_{i-1}. \qquad (\alpha)$$

We have from (α), Example 7,

$$\overline{P_{i-1}}\Big|_\mu^1 = \int_\mu^1 \mu \frac{dP_i}{d\mu} d\mu - i\int_\mu^1 P_i d\mu = \mu \overline{P_i}\Big|_\mu^1 - (i+1)\int_\mu^1 P_i d\mu;$$

$$\therefore \quad P_{i-1} = \mu P_i + (i+1)\int_\mu^1 P_i \, d\mu. \qquad (1)$$

But from the fundamental equation for P_i,

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_i}{d\mu}\right\} + i(i+1)P_i = 0,$$

we have by integration

$$i(i+1)\int_\mu^1 P_i \, d\mu = (1-\mu^2)\frac{dP_i}{d\mu}.$$

Substituting in (1), we have the result (α).

10. For an attracting body or system symmetrical about an axis, in shape and density, prove that if the Potential (external) is arranged in the series

$$\frac{\alpha_0}{r} + \alpha_1 \frac{P_1}{r^2} + \alpha_2 \frac{P_2}{r^3} + \dots + \alpha_i \frac{P_i}{r^{i+1}} + \dots$$

of Zonal Harmonics, the lines of force trace out surfaces given by the equation

$$\alpha_0 \mu - 2\frac{\alpha_1}{r}\int_\mu^1 P_1 d\mu - 3\frac{\alpha_2}{r^2}\int_\mu^1 P_2 d\mu - \dots - (i+1)\frac{\alpha_i}{r^i}\int_\mu^1 P_i d\mu \dots = C,$$

in which different constant values are assigned to C.

Let O be the origin, and P any point external to the body, OP

being r; let S be the radial attraction intensity at P (acting in the direction PO), and T the attraction intensity perpendicular to OP in the sense in which θ increases.

Then, the resultant of S and T acting along the tangent to the line of force at P, we have as the differential equation of this line

$$\frac{-dr}{rd\theta} = \frac{R}{T}. \tag{1}$$

But $$R = -\frac{dV}{dr} = \frac{1}{r^2}(\alpha_0 + 2\frac{\alpha_1}{r}P_1 + \ldots + (i+1)\frac{\alpha_i}{r^i}P_i + \ldots), \tag{2}$$

and $$S = \frac{dV}{rd\theta} = \frac{1}{r^2}(\frac{\alpha_1}{r}\frac{dP_1}{d\theta} + \ldots + \frac{\alpha_i}{r^i}\frac{dP_i}{d\theta} + \ldots). \tag{3}$$

Observing that $\frac{d}{d\theta} = -\sqrt{1-\mu^2}\frac{d}{d\mu}$, we get, by substituting from (2) and (3) in (1), the equation

$$\alpha_0 d\mu + \alpha_1\left\{2\frac{P_1}{r}d\mu + \frac{1-\mu^2}{r^2}\frac{dP_1}{d\mu}dr\right\} + \ldots$$
$$+ \alpha_i\left\{(i+1)\frac{P_i}{r^i}d\mu + \frac{1-\mu^2}{r^{i+1}}\frac{dP_i}{d\mu}dr\right\} + \ldots. \tag{4}$$

Now, by Example 9, the coefficient of α_i in this equation is

$$(i+1)\left\{\frac{P_i}{r^i}d\mu + \frac{i}{r^{i+1}}dr\int_\mu^1 P_i d\mu\right\}, \text{ that is, } -(i+1)D\left\{\frac{1}{r^i}\int_\mu^1 P_i d\mu\right\},$$

where D stands for the total differential of the quantity in brackets (with respect to μ and r). Hence, integrating (4), we have the equation which was to be proved.

In particular, if the series for the Potential stops with P_2, the equation of a line of force is

$$\alpha_0\mu - 2\frac{\alpha_1}{r}\int_\mu^1 \mu d\mu - 3\frac{\alpha_2}{r^2}\int_\mu^1 \tfrac{1}{2}(3\mu^2-1)d\mu = C,$$

or $$\alpha_0\cos\theta - \frac{\alpha_1}{r}\sin^2\theta - \frac{3}{2}\frac{\alpha_2}{r^2}\cos\theta\sin^2\theta = C.$$

11. If the density at any point of a solid sphere is proportional to the distance from a given central plane, find the Potential at any external point, P.

Ans. If a = radius of sphere, R = distance of P from centre, and $\rho = \lambda z'$ where z' is the perpendicular from any point on the plane,

$$V = \gamma \cdot \frac{4\lambda\pi a^5}{15R^3}.z.$$

[Here $V = \gamma\lambda\iiint r'^3\mu'(\frac{L_0}{R} + L_1\frac{r'}{R^2} + \ldots)dr'd\mu'd\phi'$. Integrate

first from $r' = 0$ to $r' = a$, and since μ' is a Harmonic of the first degree, the only term not vanishing is that in L_1; therefore

$$V = \frac{\gamma\lambda a^5}{5R^2}\int_{-1}^{1}\int_0^{2\pi} L_1\mu'\,d\mu'\,d\phi' = \frac{\gamma\lambda a^5}{5R^2}\cdot\frac{4\pi}{3}\mu;\ \text{\&c.}]$$

12. In the same way exactly prove that if the density at any point in a solid sphere of radius a is proportional to any solid Harmonic, S_i, of positive degree in the co-ordinates of the point, the Potential of the sphere at any external point whose distance from the centre is R is

$$\frac{4\lambda\pi a^{2i+3}}{(2i+1)(2i+3)}\cdot\frac{S_i}{R^{2i+1}},$$

the co-ordinates (x, y, z) involved in S_i being those of the given external point, and λ being the constant involved in the density.

Deduce the result also for a spherical shell and any internal point.

13. If the origin of co-ordinates is transferred from O to a point O' along the axis of z (from which θ is measured), calculate the solid Zonal Harmonic of degree i with reference to O' as origin in terms of the solid Zonal Harmonics with reference to O.

Let Z_i be the solid Harmonic of degree i with reference to O, and Z'_i that with reference to O'. Then, with the notation of Art. 329,

$$Z_i = f(z, \zeta);\ \text{and if } OO' = h,\ Z_i' = f(z+h, \zeta);$$

$$\therefore\ Z_i' = Z_i + h\frac{dZ_i}{dz} + \frac{h^2}{1\,.\,2}\frac{d^2Z_i}{dz^2} + \dots \tag{1}$$

[Here (z, ζ) are the cylindrical co-ordinates of a point P with reference to O, and $(z+h, \zeta)$ are the co-ordinates of the same point, P, with reference to O'.]

Now $Z_i = r^i P_i$, where P_i is the surface Zonal Harmonic, and $\frac{d}{dz} = \frac{1-\mu^2}{r}\frac{d}{d\mu} + \mu\frac{d}{dr}$. Hence

$$\frac{dZ_i}{dz} = r^{i-1}\left\{(1-\mu^2)\frac{dP_i}{d\mu} + i\mu P_i\right\}$$

$$= ir^{i-1}\,.\,P_{i-1}\ \text{(by Example 9)}$$

$$= iZ_{i-1}.$$

Hence, again, $\frac{d^2Z_i}{dz^2} = i(i-1)Z_{i-2}$, &c., and therefore

$$Z_i' = Z_i + ihZ_{i-1} + \frac{i(i-1)}{1\,.\,2}h^2Z_{i-2} + \dots ih^{i-1}Z_1 + h^n.$$

Let us find in the same way the value of the Solid Harmonic of negative degree, $-(i+1)$. Let this Harmonic, with reference to O, be U_i, or $\frac{P_i}{r^{i+1}}$.

Then
$$\frac{dU}{dz} = \frac{1}{r^{i+2}}\left\{(1-\mu^2)\frac{dP_i}{d\mu} - (i+1)\mu P_i\right\}$$
$$= \{iP_{i-1} - (2i+1)\mu P_i\}/r^{i+2} \quad \text{(by Example 9)}$$
$$= -(i+1)P_{i+1}/r^{i+2} \quad \text{(by Example 8)}$$
$$= -(i+1)U_{i+1}.$$

Hence $U_i' = U_i - (i+1)hU_{i+1} + \frac{(i+1)(i+2)}{1\,.\,2}h^2U_{i+2} - ad\ infin.$

14. Arrange the expression $\frac{\cos^2\theta}{c^2} + \frac{\sin^2\theta\cos^2\phi}{a^2} + \frac{\sin^2\theta\cos^2\phi}{b^2}$ as a series of Spherical Harmonics.

Ans. $\frac{1}{3}\left(\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1}{3}\left(\frac{1}{c^2}-\frac{1}{2a^2}-\frac{1}{2b^2}\right)(3\mu^2-1)$
$$+\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)(1-\mu^2)\cos 2\phi.$$

15. Express the central radius vector of a nearly spherical ellipsoid by Spherical Harmonics.

Ans. If $a-c = ck$, $b-c = ck'$, we have
$$r = c\left\{1+\tfrac{1}{3}(k+k')-\tfrac{1}{6}(k+k')(3\mu^2-1)-\tfrac{1}{2}(k'-k)(1-\mu^2)\cos 2\phi\right\},$$
which is of the form $r = c(Y_0 + Y_2)$.

16. If the expression $(1-2\mu x+x^2)^{-\frac{2m+1}{2}}$ be developed in a series in the form $Q_0 + Q_1x + Q_2x^2 + \ldots + Q_ix^i + \ldots$, prove that, in analogy with the Legendrian coefficients,
$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} - 2m\mu\frac{dQ_i}{d\mu} + i(2m+i+1)Q_i = 0.$$

Differentiate the given identity (α) with regard to x, and we obtain an identity (β); differentiate (α) with regard to μ, and we obtain an identity (γ); from (β) and (γ) we have, by equating the coefficients of x^i,
$$iQ_i = \mu\frac{dQ_i}{d\mu} - \frac{dQ_{i-1}}{d\mu}. \tag{δ}$$

Multiply (α) by $(2m+1)(\mu-x)$ and (β) by $1-2\mu x+x^2$, and we have
$$(i+1)Q_{i+1} - (2m+2i+1)\mu Q_i + (2m+i)Q_{i-1} = 0. \tag{ϵ}$$

Differentiating (ϵ) with regard to μ and eliminating Q_{i-1} by (δ),
$$\frac{dQ_{i+1}}{d\mu} - \mu\frac{dQ_i}{d\mu} - (2m+i+1)Q_i = 0. \tag{ζ}$$

Replace i by $i+1$ in (δ), combine with (ζ), and we have
$$(i+1)Q_{i+1} = -(1-\mu^2)\frac{dQ_i}{d\mu} + (2m+i+1)\mu Q_i. \tag{η}$$

Differentiate (η) with respect to μ, and subtract the result from (ζ) multiplied by $(i+1)$, and we have the required equation.

17. In this development how far is it true that

$$\int_{-1}^{1} Q_i Q_{i'} d\mu = 0,$$

i and i' being different integers?

It is always true if one of the numbers i, i' is even and the other odd. In Green's equation, applied through the interior and over the surface of a sphere, let $U = Q_i$, $V = Q_{i'}$, and observe that

$$\nabla^2 Q_i = \frac{1}{r^2}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} = \frac{1}{r^2}\left\{2m\mu\frac{dQ_i}{d\mu} - i(2m+i+1)Q_i\right\}.$$

18. Exhibit $\cos^2\theta\sin^2\theta\sin\phi\cos\phi$ as a series of Surface Harmonics.

The simplest method is, of course, that given at the end of Art. 352, which deduces the result by expressing xyz^2 in the form $S_4 + r^2 S_2$. Performing ∇^2, we get $S_2 = \frac{1}{7}xy$, and $\therefore$ $S_4 = xy\,(z^2 - \frac{1}{7}r^2)$. Now $S_4 = r^4 Y_4$ and $S_2 = r^2 Y_2$; whence Y_4 and Y_2 are at once found. Nevertheless it may be useful to show how to deduce the result by means of Laplacians and integration.

The given function is $\frac{1}{2}\mu^2(1-\mu^2)\sin 2\phi$, and since the term of highest degree in the quantities μ, $\sqrt{1-\mu^2}\sin\phi$, and $\sqrt{1-\mu^2}\cos\phi$ is obviously of the fourth degree in these quantities, it follows that the given function must be of the form $Y_1 + Y_2 + Y_3 + Y_4$, the term in Y_0 being obviously non-existent.

Now Y_1 is obtained by taking $\frac{1}{2}\mu'^2(1-\mu'^2)\sin 2\phi'$, multiplying it by L_1, and integrating $\left.\overline{\mu'}\right|_{-1}^{1}$, $\left.\overline{\phi'}\right|_{0}^{2\pi}$; but as L_1 is of the form $A + B\cos(\phi-\phi')$, and as $\int_0^{2\pi}\sin 2\phi'\cos(\phi-\phi')\,d\phi' = 0$, we see at once that $Y_1 = 0$.

It is clear that in L_2, L_3, L_4, it is only the terms involving $\cos 2(\phi-\phi')$ that will give any result when multiplied by $\sin 2\phi'$ and integrated. In L_2 occurs the term (see table, p. 252)

$$\tfrac{3}{4}(1-\mu^2)(1-\mu'^2)\cos 2(\phi-\phi');\ \text{hence (Art. 353)}$$

$$\frac{1}{2}\cdot\frac{3}{4}\int_{-1}^{1}\int_0^{2\pi}(1-\mu^2)\,.\,\mu'^2(1-\mu'^2)^2\sin 2\phi'\cos 2(\phi-\phi')\,d\mu'\,d\phi' = \frac{4\pi}{5}Y_2.$$

But $\int_0^{2\pi}\sin 2\phi'\cos 2(\phi-\phi')\,d\phi' = \pi\sin 2\phi$; and

$$\int_{-1}^{1}\mu'^2(1-\mu'^2)^2 d\mu' = \frac{16}{3\,.\,5\,.\,7},$$

$$\therefore\quad Y_2 = \tfrac{1}{14}(1-\mu'^2)\sin 2\phi.$$

Again, in L_3 the only possible term is

$$\tfrac{15}{4}\mu(1-\mu^2)\mu'(1-\mu'^2)\cos 2(\phi-\phi');$$

but the integration in μ' destroys this, since it gives

$$\int_{-1}^{1} f(\mu'^2) \,.\, \mu' \, d\mu',$$

which obviously vanishes. Hence $Y_3 = 0$.

Finally, in L_4 the only term to be taken is

$$\tfrac{5}{16}(1-\mu^2)(7\mu^2-1) \,.\, (1-\mu'^2)(7\mu'^2-1).$$

Hence

$$\tfrac{1}{2} \cdot \tfrac{5}{16}(1-\mu^2)(7\mu^2-1)\int_{-1}^{1} \mu'^2(1-\mu'^2)^2(7\mu'^2-1)\, d\mu'$$
$$\times \int_0^{2\pi} \sin 2\phi' \cos 2(\phi-\phi')\, d\phi' = \frac{4\pi}{9} Y_4;$$

$$\therefore \quad Y_4 = \tfrac{1}{14}(1-\mu^2)(7\mu^2-1)\sin 2\phi;$$

and therefore

$$\cos^2\theta \sin^2\theta \sin\phi \cos\phi = \tfrac{1}{14}(1-\mu^2)\sin 2\phi + \tfrac{1}{14}(1-\mu^2)(7\mu^2-1)\sin 2\phi,$$

which is of the form $Y_2 + Y_4$.

19. Exhibit $\cos\theta \sin^3\theta \cos^2\phi \sin\phi$ as a series of Harmonics.

Ans. $\{\tfrac{1}{7}\mu\sqrt{1-\mu^2}\sin\phi\} + \{-\tfrac{1}{28}\sqrt{1-\mu^2}(7\mu^3-3\mu)\sin\phi$
$+ \tfrac{1}{4}\mu(1-\mu^2)^{\frac{3}{2}}\sin 3\phi\}$,

which is of the form $Y_2 + Y_4$.

[The given expression is $\tfrac{1}{4}\mu(1-\mu^2)^{\frac{3}{2}}(\sin\phi + \sin 3\phi)$; hence the only terms to attend to in the L's are those in $\cos(\phi-\phi')$ and $\cos 3(\phi-\phi')$. The term in L_1 is destroyed by the integration in μ', which also destroys both the terms in L_3.] Deduce the result also from x^2yz.

20. Why cannot $\sin\theta$, $\sin^3\theta$, or any odd power of $\sin\theta$ be expanded in a finite series of Harmonics?

Because they are of the form $(1-\mu^2)^{\frac{2n+1}{2}}$, which can be developed in an infinite series ascending by powers of μ, and every term, such as μ^m, can be developed in a finite series of Zonals, P_1, P_2, ... Also a function can be expanded in only one way.

21. Prove that $$\int_{-1}^{1} P_n^2\, d\mu = 2/(2n+1).$$

If O is a point within a circle of radius a and centre C, and if P is a point on the circle, then, c denoting CO, r OP, θ the angle OCP, and $z = \exp. i\theta$, we have

$$r = (a^2 - 2ac\cos\theta + c^2)^{\frac{1}{2}} \qquad \text{(A)}$$
$$= a(1 - cz/a)^{\frac{1}{2}}(1 - c/az)^{\frac{1}{2}}$$

and $$\log r = \log a - (f\cos\theta + \tfrac{1}{2}f^2\cos 2\theta + \ldots \frac{1}{n}f^n\cos n\theta + \ldots),$$ where $c = af$.

Hence $dr/d\theta = r(f\sin\theta + f^2\sin 2\theta + \ldots + f^n \sin n\theta + \ldots)$.

Now $$dr/d\theta = (ac\sin\theta)/r \text{ by (A)},$$

and $$1/r = (1/a)(1 + P_1 f + P_2 f^2 + \ldots + P_n f^n + \ldots).$$

$$\therefore\; f(1 + P_1 f + P_2 f^2 + \ldots P_n f^n + \ldots)^2$$
$$= f + f^2 \sin 2\theta/\sin\theta + \ldots + f^n \sin n\theta/\sin\theta + \ldots.$$

Equate the coefficients of f^{2n} on both sides

$$\therefore\quad P_n^2 + 2P_1 P_{2n-1} + 2P_2 P_{2n-2} + 2P_3 P_{2n-3} + \ldots = \sin(2n+1)\theta/\sin\theta,$$

and, since $\displaystyle\int_{-1}^{1} P_i P_{2n-i}\, d\mu = 0$, and $d\mu = -\sin\theta\, d\theta$,

$$\therefore\quad \int_{-1}^{1} P_n^2\, d\mu = \int_{\pi}^{0} \frac{\sin(2n+1)\theta}{\sin\theta}(-\sin\theta\, d\theta)$$
$$= \int_{0}^{\pi} \sin(2n+1)\theta\, d\theta = 2/(2n+1).$$

22. Prove that

$$n\int_{1}^{\mu} (1-\mu^2)^{\frac{1}{2}n-1} P_n\, d\mu = -(1-\mu^2)^{\frac{1}{2}n} P_{n-1}.$$

If $D \equiv d/d\mu$, we have by Examples 8 and 9

$$(1-\mu^2)DP_n = n(P_{n-1} - \mu P_n), \tag{1}$$
$$nP_n - (2n-1)\mu P_{n-1} + (n-1)P_{n-2}. \tag{2}$$

From (1) $\quad (1-\mu^2)DP_{n-1} = (n-1)(P_{n-2} - \mu P_{n-1}).$

$$\therefore\; (1-\mu^2)DP_{n-1} - \mu n P_{n-1} = (n-1)P_{n-2} - (2n-1)\mu P_{n-1}$$
$$= -nP_n \text{ by (2)}.$$

$$\therefore\; (1-\mu^2)^{\frac{1}{2}n} DP_{n-1} - n\mu(1-\mu^2)^{\frac{1}{2}n-1} P_{n-1} = -n(1-\mu^2)^{\frac{1}{2}n-1} P_n,$$

and the left-hand side is a perfect differential.

355.] **Case of Spheroids.** Any solid body differing little in shape from a sphere is called a *Spheroid*. Supposing the body to be homogeneous, the radius vector from its centre of mass to any point on its surface will be nearly of constant length. Thus (following the notation of Laplace), if α denote a small numerical quantity, and R' any radius vector from the centre of mass to the surface, we shall have

$$R' = a + \alpha a f(\mu', \phi'), \tag{1}$$

where a is a constant length and $f(\mu', \phi')$ some function of the angular co-ordinates depending on the precise shape of the bounding surface. Laplace uses y' for the function $f(\mu', \phi')$, and

he assumes that y' is expanded in a series of Spherical Harmonics; thus,

$$R' = a + \alpha a\,(Y_0' + Y_1' + Y_2' + \ldots + Y_i' + \ldots). \tag{2}$$

If the series stops with Y_2', the bounding surface will be that of an ellipsoid.

External Point. To calculate the Potential at an external point, P, produced by a homogeneous spheroid, the distance of P from the origin O being greater than the greatest radius vector from O to the surface, let $OP = R$, $\rho =$ density of the body, and (r', μ', ϕ') the co-ordinates of any point P' in the body of the spheroid. Then, γ being, as usual, the gravitation constant,

$$V = \gamma \int_0^{2\pi}\int_{-1}^{1}\int_0^{R'} \frac{\rho r'^2 dr' d\mu' d\phi'}{PP'} \tag{3}$$

$$= \frac{\gamma}{R}\int_0^{2\pi}\int_{-1}^{1}\int_0^{R'} \Big(L_0 + L_1\frac{r'}{R} + L_2\frac{r'^2}{R^2} + \ldots$$

$$+ L_i\frac{r'^i}{R^i} + \ldots\Big)\rho r'^2 dr' d\mu' d\phi' \tag{4}$$

$$= \frac{\gamma\rho}{R}\int_0^{2\pi}\int_{-1}^{1}\Big(\tfrac{1}{3}L_0 R'^3 + L_1\frac{R'^4}{4R} + \ldots + L_i\frac{R'^{i+3}}{(i+3)R^i} + \ldots\Big)d\mu' d\phi'.$$

Now from (2), neglecting higher powers than the first of α,

$$R'^{i+3} = a^{i+3}\{1 + (i+3)\alpha(Y_0' + Y_1' + \ldots + Y_i' + \ldots)\},$$

and by substitution in the last value of V, since it is (Art. 351) only the term $\int\int L_i Y_i' d\mu' d\phi'$ which does not vanish, we have

$$V = \gamma\frac{4\pi\rho a^3(1 + 3\alpha Y_0)}{3R}$$

$$+ \alpha\gamma.\frac{\rho a^3}{R}\int_0^{2\pi}\int_{-1}^{1}\Big(\frac{a}{R}L_1 Y_1' + \frac{a^2}{R^2}L_2 Y_2' + \ldots + \frac{a^i}{R^i}L_i Y_i' + \ldots\Big)d\mu' d\phi'.$$

Now the volume of the Spheroid is $\frac{4}{3}\pi a^3(1 + 3\alpha Y_0)$, and if we choose a so that $\frac{4}{3}\pi a^3$ shall be the volume, Y_0 will be zero. Thus, attending to the result (η), Art. 352, we have

$$V = \gamma\frac{M}{R} + 3\alpha\gamma\frac{M}{R}\Big(\frac{a}{3R}Y_1 + \frac{a^2}{5R^2}Y_2 + \ldots + \frac{a^i}{(2i+1)R^i}Y_i + \ldots\Big), \tag{5}$$

where M is the mass of the Spheroid.

It is very easy to see that, with the origin at the centre of mass of the Spheroid, the term Y_1 is zero.

For if, in general, $(\bar{x}, \bar{y}, \bar{z})$ are the co-ordinates of the centre of mass, we have

$$M\bar{x} = \rho\int\int\int r'^3 (1-\mu'^2)^{\frac{1}{2}} \cos\phi' \, dr' \, d\mu' \, d\phi'$$
$$= \tfrac{1}{4}\rho\int\int R'^4 (1-\mu'^2)^{\frac{1}{2}} \cos\phi' \, d\mu' \, d\phi', \tag{6}$$

and since $(1-\mu'^2)^{\frac{1}{2}}\cos\phi'$ is a Spherical Harmonic of the first degree, in the expansion of R'^4—viz. $a^4\{1+4\alpha(Y_1'+Y_2')+\ldots\}$ —the only term that will not identically vanish in (6) is

$$\alpha\rho a^4\int\int Y_1' (1-\mu'^2)^{\frac{1}{2}} \cos\phi' \, d\mu' \, d\phi'.$$

But this is zero because $\bar{x} = 0$. Hence

$$\int\int Y_1' (1-\mu'^2)^{\frac{1}{2}} \cos\phi' \, d\mu' \, d\phi' = 0. \tag{7}$$

Similarly, since $\bar{y} = 0$, we must have

$$\int\int Y_1' (1-\mu'^2)^{\frac{1}{2}} \sin\phi' \, d\mu' \, d\phi' = 0\,; \tag{8}$$

and since $\bar{z} = 0$, $\quad \int\int Y_1' \mu' \, d\mu' \, d\phi' = 0. \quad (9)$

But Y_1' is (Art. 350) of the form

$$A\mu' + (1-\mu'^2)^{\frac{1}{2}} (B\cos\phi' + C\sin\phi'),$$

where A, B, C are constants, and the results (7), (8), (9) make $A = B = C = 0$, as is easily seen either by direct integration, or by multiplying the left-hand sides of these equations by A, B, C and adding. We thus get $\int\int Y_1'^2 d\mu' \, d\phi' = 0$, which requires Y_1' to vanish identically.

For example, take the case of a nearly spherical ellipsoid of revolution round the smaller axis, c.

In this case (see Example 15, p. 261) $k = k'$, and we have

$$R' = c\{1 + \tfrac{2}{3}k - \tfrac{1}{3}k(3\mu'^2 - 1)\}.$$

But the a in (5) is determined from the equation

$$a^3 = c^3(1+2k)\,; \qquad \therefore \quad a = c\,(1+\tfrac{2}{3}k),$$

and since $R' = a\,(1+\alpha Y_2')$, we have $\alpha = -\frac{1}{3}k$. Hence (5) gives

$$V = \gamma\frac{M}{R} - \tfrac{1}{5}\gamma k M \frac{a^2}{R^3}(3\mu^2 - 1),$$

in which a or c may be used indifferently in the small term.

If the Spheroid is not homogeneous, but consists of strata of different densities, each stratum differing but little from a sphere, the Potential can still be very easily expressed. Thus, let $r' = a'(1+\alpha y')$ be the equation of any stratum, a' being the radius of a sphere whose volume is equal to that of the stratum, so that

$$y' = Y_1' + Y_2' + \ldots + Y_i' + \ldots,$$

where the Y''s involve a' as well as μ' and ϕ', unless the strata are all similar.

Now if the Spheroid were homogeneous and of density ρ as far as the stratum a', the Potential of this portion would be given by the equation

$$\frac{V}{\gamma} = \frac{4}{3}\frac{\pi}{R} \cdot \rho a'^3 + \frac{4\alpha\pi}{R}\left\{\frac{1}{3R}\rho Y_1' a'^4 + \frac{1}{5R^2}\rho Y_2' a'^5 + \dots \right.$$
$$\left. \dots + \frac{1}{(2i+1)R^i}\rho Y_i' a'^{i+3} + \dots\right\}.$$

Let $a' + da'$ be the constant of the next stratum outside, and let the value of V due to the whole portion of the Spheroid, supposed homogeneous and still of density ρ, up to and including this stratum, be written down. Subtract the first result from the second and we obtain the Potential due to the shell of density ρ included between the strata a' and $a' + da'$.

The Potential of the homogeneous solid $a' + da'$ being $V + dV$, we have by subtracting that due to the homogeneous solid a',

$$\frac{dV}{\gamma} = \frac{4\pi}{3R}\rho d(a'^3) + \frac{4\alpha\pi}{R}\rho\left\{\frac{1}{3R}d(a'^4 Y_1') + \dots \right.$$
$$\left. \dots + \frac{1}{(2i+1)R^i} d(a'^{i+3} Y_i') + \dots\right\},$$

the independent variable in the differentiations being a', the parameter which determines any one stratum of constant density. Now if the value of a' for the bounding surface of the Spheroid is a, we have by integrating the above

$$\frac{V}{\gamma} = \frac{4\pi}{3R}\int_0^a \rho d(a'^3) + \frac{4\alpha\pi}{R}\int_0^a \rho\left\{\frac{1}{3R}d(a'^4 Y_1') + \dots \right.$$
$$\left. \dots + \frac{1}{(2i+1)R^i} d(a'^{i+3} Y_i') + \dots\right\},$$

or
$$\frac{V}{\gamma} = \frac{M}{R} + \frac{4\alpha\pi}{3R^2}\int_0^a \rho d(a'^4 Y_1') + \dots$$
$$\dots + \frac{4\alpha\pi}{(2i+1)R^{i+1}}\int_0^a \rho d(a'^{i+3} Y_i') + \dots. \quad (10)$$

Internal Point. If the point, P, at which the value of the Potential is desired, is inside the Spheroid, we may treat the spheroid as consisting of a sphere and a superficial layer which is everywhere of comparatively small thickness.

The Potential of a solid homogeneous sphere at an internal point has been already found. We must therefore find the Potential at P due to the shell at the surface of this sphere—observing that, according to the shape of the spheroid, the thickness of this shell measured outwards from the surface of the sphere may be positive or negative. If the equation of the surface is $r = a(1+\alpha y)$, the thickness of the shell at any point is (nearly) $\alpha a y$; or the value of r' ranges from $r' = a$ to $r' = a(1+\alpha y)$. If v is the Potential at P (internal) due to the shell,

$$\frac{v}{\gamma} = \iiint \rho \left(L_0 + L_1 \frac{R}{r'} + \ldots + L_i \frac{R^i}{r'^i} + \ldots\right) r' dr' d\mu' d\phi'.$$

Performing the integration in r' first, we have

$$\frac{v}{\gamma} = \alpha\rho \iint \left(L_0 a^2 + L_1 R a + \ldots + L_i \frac{R^i}{a^{i-2}} + \ldots\right) y' d\mu' d\phi',$$

which, by Art. 352, is

$$\frac{v}{\gamma} = 4\alpha\pi\rho a^2 \left\{ Y_0 + \frac{R}{3a} Y_1 + \frac{R^2}{5a^2} Y_2 + \ldots + \frac{R^i}{(2i+1)a^i} Y_i + \ldots \right\}, \quad (11)$$

in which the Y's belong to the attracted point P. To this must be added $2\pi\rho a^2 - \frac{2}{3}\pi\rho R^2$, which is due to the sphere of radius a, so that

$$\frac{V}{\gamma} = 2\pi\rho a^2 - \tfrac{2}{3}\pi\rho R^2 + 4\alpha\pi\rho a^2 \left\{ \ldots + \frac{R^i}{(2i+1)a^i} Y_i + \ldots \right\}. \quad (12)$$

As has been already proved, the terms Y_0 and Y_1 may be dispensed with.

The case of a heterogeneous spheroid is treated exactly as before. The point P being internal, let b be the parameter of the stratum of constant density passing through P, and take for V the sum of the Potentials due to the spheroid as far as this stratum and to the portion between this stratum and the bounding surface (of parameter a). The point P is external to the first, and the corresponding part of V is given by (10) in which we have simply to change the limit a to b in the integrations. The Potential due to any stratum (a', ρ) surrounding P can be obtained by subtracting the Potential due to a *solid* homogeneous spheroid, (a', ρ) from that due to a *solid* homogeneous spheroid

$(a'+da', \rho)$. Thus by (12) the Potential due to the *stratum* (a', ρ) is

$$2\pi\rho d(a'^2) + 4\alpha\pi\rho d\left\{a'^2 Y_0' + \frac{R}{3}a' Y_1' + \ldots + \frac{R^i}{2i+1}\frac{Y_i'}{a'^{i-2}} \&c.\right\}.$$

Integrating this between $a' = b$ and $a' = a$, we have by addition to the first portion,

$$\frac{V}{\gamma} = \frac{4\pi}{3R}\int_0^b \rho d(a'^3) + \ldots + \frac{4\alpha\pi}{(2i+1)R^{i+1}}\int_0^b \rho d(a'^{i+3}Y_i') + \ldots$$
$$+ 2\pi\int_b^a \rho d(a'^2) + \ldots + \frac{4\alpha\pi}{2i+1}R^i\int_b^a \rho d\left(\frac{Y_i'}{a'^{i-2}}\right) + \ldots \quad (13)$$

For the discussion of the figure and law of density of the strata of the earth the reader will, of course, consult the *Mécanique Céleste*. A valuable epitome of Laplace's and other results will be found in Pratt's *Treatise on Attractions, Laplace's Functions, and the Figure of the Earth.*

Miscellaneous Examples.

1. Find the work required to scatter the particles of a uniform circular plate to infinite distances from each other (for the law of nature).

Ans. Let M be the mass of the plate in grammes, a its radius in centimetres, and γ the C.G.S. constant of gravitation; then the work is

$$\frac{8\gamma M^2}{3\pi a} \text{ ergs.}$$

At any distance, x, from the centre, inside the plate

$$V = 4\gamma\rho\tau\int_0^{\frac{\pi}{2}} \sqrt{a^2 - x^2\sin^2\theta}\, d\theta,$$

where τ = thickness of plate. Hence

$$\tfrac{1}{2}\int V\,dm = 4\pi\gamma\rho^2\tau^2\int_0^{\frac{\pi}{2}}\int_0^a \sqrt{a^2 - x^2\sin^2\theta}\,.\,x\,dx\,d\theta.$$

Perform the integration in x first; &c.

2. Considering the attraction-intensity of an infinite plate at a point near its surface, show that it is greater for the law of inverse square than for the law $1/r^n$ when $n < 2$, and less for the law of inverse square than for the law $1/r^n$ when $n > 2$.

The attracted particle having any position on the axis of the plate (assumed circular), the attraction-intensity for the law $1/r^n$ is

$$2\pi\gamma\rho\tau\frac{(1-\cos^n a)\tan^{n-2} a}{n-1}.$$

If the particle is near the plate, $\cos a = x$, where x is very small, $\tan a = 1/x$, and the most important part of this expression becomes $\frac{1}{(n-1)x^{n-2}}$; from which the result follows.

3. At a point in the plane of a uniform circular plate outside its circumference, the Potential is

$$4\gamma\rho\tau\left(E - \frac{x^2-a^2}{x^2}K\right).x,$$

where x is the distance of the point from the centre, and K and E are the complete elliptic integrals of the first and second kinds with modulus a/x.

[Let P be the point, O the centre, Q any point on the circumference, $\angle OPQ = \theta$; then

$$V = 4\gamma\rho\tau\int_0^{\alpha}\sqrt{a^2-x^2\sin^2\theta}\,d\theta,$$

where $\alpha = \sin^{-1}(a/x)$. Let $x\sin\theta = a\sin\phi$, where ϕ is the angle between QP and QO; &c.]

4. Find a function, ϕ, of r only which satisfies the equation

$$(\nabla^2+a^2)\phi = 0,$$

where a is independent of r.

Ans. $\phi = A\frac{e^{ar}}{r} + B\frac{e^{-ar}}{r}$.

The equation (γ), p. 176, becomes $\frac{d^2.r\phi}{dr^2} + a^2.r\phi = 0$.

5. Fig. 228, p. 1, represents a homogeneous solid rectangular block whose density is ρ grammes per cub. cm.; the sides are $AD = 2a$ cm., $BD = b$ cm., $DO' = h$ cm.; find the attraction-intensity at a point, P, which is on the perpendicular to AD at its middle point and lies in the plane of the face $AOBD$.

Ans. If p is the distance (in centimetres) of P from the side AD, and if X, Z are the components of the force-intensity in and perpendicular to the plane $AOBD$,

$$X = 2\gamma\rho\int_p^{p+b}\sin^{-1}\frac{ah}{\sqrt{(a^2+x^2)(h^2+x^2)}}.dx; \quad \text{(dynes per gramme.)}$$

$$Z = 2\gamma\rho\int_p^{p+b}\log_e\frac{\sqrt{h^2+x^2}\,(a+\sqrt{a^2+x^2})}{x(a+\sqrt{h^2+a^2+x^2})}.dx. \quad \text{(dynes per gramme.)}$$

6. Apply the preceding to calculate the deviation of the plumb-line caused by a large rectangular table-land in the following instance.

'A table-land 1,610 feet high, commencing at a distance of 20 miles from Takal K'hera near the great arc of meridian in India, runs

80 miles north, and 60 miles to the east and 60 to the west.' (Pratt's *Attractions, Laplace's Functions, and the Figure of the Earth*, p. 48.)

Observe that h is here very small compared with the other linear dimensions.

Assume ρ to be 2·8, i.e. about half the mean density of the Earth, or the density of statuary marble; also assume 160,933 centimetres in 1 mile. Then, since a gramme mass weighs at the surface of the Earth about 980 dynes, the circular measure of the deviation is $\frac{X}{980}$; and the deviation is found to be about 4·8″—so considerable a disturbance that (it is stated) the place in question was abandoned as a principal station of the survey. We have neglected Z in this result, as is, of course, allowable.

7. When by the method of Inversion (Art. 331) a system of points (x', y', z') is deduced from a given system (x, y, z) show that if the operations $x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}$, or $r\frac{d}{dr}$, and $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$ are, respectively, denoted by δ and ∇^2, we have

$$\delta' = -\delta,$$

$$\nabla'^2 = \frac{r^4}{k^4}\nabla^2 - \frac{2}{k^2}\delta.$$

NOTE.

A.

Potential of a Homogeneous Ellipsoid.

The following investigation of the Potential produced at an external point, P, by a homogeneous solid ellipsoid has been given by Colonel A. R. Clarke (see the *Phil. Mag.*, December, 1877).

Take the principal axes of the ellipsoid as axes of co-ordinates; let x, y, z be the co-ordinates of P; let Q be any point inside the mass at which an element dm is taken; let x', y', z' be the co-ordinates of Q; O the centre of the ellipsoid, $OP = R$, $OQ = r$, and $\psi = \cos POQ$. Then, ρ being the mass per unit volume of the body,

$$V = \gamma\rho\int\frac{dm}{(R^2-2Rr\,.\,\psi+r^2)^{\frac{1}{2}}} \tag{1}$$

$$= \frac{\gamma\rho}{R}\int\left(1+P_1\frac{r}{R}+P_2\frac{r^2}{R^2}+P_3\frac{r^3}{R^3}+\ldots\right)dm, \tag{2}$$

where $P_1, P_2, \ldots$ are the Legendrian coefficients, as in (γ), p. 243, or at p. 252, with ψ written instead of μ.

But since P_1, P_3, P_5,... are each of the form $\psi f(\psi^2)$, the terms in them vanish because of the complete symmetry of the figure, so that

$$V = \frac{\gamma\rho}{R}\int\left(1 + P_2\frac{r^2}{R^2} + P_4\frac{r^4}{R^4} + P_6\frac{r^6}{R^6} + \dots\right)dm. \tag{3}$$

Now $\psi = \dfrac{xx' + yy' + zz'}{rR} = \dfrac{lx' + my' + nz'}{r} = \dfrac{\Delta}{r}$, suppose, l, m, n being the direction-cosines of OP. Hence from the values of the Legendrians, p. 252, we have

$$P_2 r^2 = \tfrac{1}{2}(3\Delta^2 - r^2);\ P_4 r^4 = \tfrac{1}{8}(35\Delta^4 - 30r^2\Delta^2 + 3r^4);$$
$$P_6 r^6 = \tfrac{1}{16}(231\Delta^6 - 315r^2\Delta^4 + 105r^4\Delta^2 - 5r^6).$$

The results of performing the integrations in (3) as far as $\int P_6 r^6 dm$ are very remarkable.

Thus, it will be found that if a, b, c are the semi-axes and Ω the whole volume of the ellipsoid, and if we put

$$b^2 - c^2 = d_1^2;\ c^2 - a^2 = d_2^2;\ a^2 - b^2 = d_3^2,$$

and also denote by L_2 the value of the Legendrian P_2 when l is put for μ; by M_2 the value of P_2 when m is put for μ; and by N_2 the value of P_2 when n is put for μ; with similar meanings of L_4, M_4, N_4 with reference to P_4, &c., we shall have

$$\int P_2 r^2 dm = -\frac{\Omega}{15}\{L_2(d_2^2 - d_3^2) + M_2(d_3^2 - d_1^2) + N_2(d_1^2 - d_2^2)\},$$

$$\int P_4 r^4 dm = -\frac{3\Omega}{35}\{L_4 d_2^2 d_3^2 + M_4 d_3^2 d_1^2 + N_4 d_1^2 d_2^2\},$$

$$\int P_6 r^6 dm = \frac{\Omega}{42}\{L_6 d_2^2 d_3^2(d_2^2 - d_3^2) + M_6 d_3^2 d_1^2(d_3^2 - d_1^2)$$
$$+ N_6 d_1^2 d_2^2(d_1^2 - d_2^2) + K d_1^2 d_2^2 d_3^2\},$$

where in the last $K = \dfrac{231}{16}(l^2 - m^2)(m^2 - n^2)(n^2 - l^2)$.

In these expressions for the terms in (3) the sequence is, as Colonel Clarke observes, remarkable, 'and suggests the idea that possibly an expression might be obtained for the general term.'

Attention may be called to three recent Papers:

(i) The Attraction of Ellipsoidal Shells and of Solid Ellipsoids, &c., by A. Gray, *Phil. Mag.*, April, 1907.

(ii) The Potential of a Uniform Convex Solid possessing a Plane of Symmetry, with Application to the Direct Integration of the Potential of a Uniform Ellipsoid, by S. Brodetsky, London Mathematical Society, December 11, 1913.

(iii) On the Attractions of Spherical and Ellipsoidal Shells, by A. Gray, Edinburgh Mathematical Society, June 13, 1914.

APPENDIX.

EXAMPLES.

90. $ABCD$ is a parallelogram, and O any point in its plane; find the resultant of forces represented by (1) OA, BO, OC, DO; (2) OA, OB, CO, DO.

Result. (i) Zero; (ii) $2OH$, where OH is parallel and equal to DA.

91. Forces 2 lb. wt., 8 lb. wt., and 9 lb. wt. act at a point O parallel to the sides AB, BC, and CA respectively of a triangle ABC. If $AB = 3$ in., $BC = 4$ in., and $CA = 5$ in., find the magnitude of the resultant and the tangent of the angle which its direction makes with AB. *Result.* $\frac{1}{5}\sqrt{305}$ lb. wt.; $-4/17$.

92. $ABCDEF$ is a regular hexagon of which O is the centre. Forces equal to 3, 2, 5, 1, P and Q lb. wt. act along OA, OB, OC, OD, OE, and OF respectively. Find the magnitudes of P and Q in order that the system may be in equilibrium. *Result.* $P = 4$; $Q = 3$.

93. Any arbitrary straight line cuts OP, OQ, OR, which represent three forces in magnitude, direction, and position, in A, B, C respectively, and for any two positions of the straight line

$$OP(OA)^{-1} + OQ(OB)^{-1} + OR(OC)^{-1} = 0;$$

show that the forces are in equilibrium.

Let the resultant K of OQ, OR meet ABC, $A'B'C'$ in A_1, A_1'. Prove $OA/OA_1 = OA'/OA_1'$. Since ABC, $A'B'C'$ are arbitrary, A and A_1 are coincident, and so A' and A_1'.

94. Forces of magnitudes 1, 4, 3, 5, 2, 6 act along the sides AB, BC, CD, DE, EF, FA of a regular hexagon. Find the magnitude of their resultant, and show that the distance of its line of action from the centre of the hexagon is to the length of a side as 21 : 2.

95. Forces P, $2P$, $3P$, $4P$ act along the sides AB, BC, CD, DA respectively of a square. Find the magnitude of the resultant and the point in which its line of action meets AB.

96. Forces $k\,.\,AB$, $k\,.\,BC$, $k\,.\,CD$, $k\,.\,DA$ act along the sides AB, BC, CD, DA of the parallelogram $ABCD$ in the directions AB, CB, CD, AD. Show that they are in equilibrium.

97. O is the orthocentre of a triangle ABC and AO, BO, CO meet BC, CA, AB respectively in D, E, F. Forces proportional to AO, BO, CO act along these lines and forces P, Q, R along EF, FD, DE respectively. Show that if the system is in equilibrium

$$P : Q : R :: \sin^2 B - \sin^2 C : \sin^2 C - \sin^2 A : \sin^2 A - \sin^2 B.$$

98. The moment of a force P_r about a fixed point O is G_r and the moment of the resultant P of the forces $P_1, P_2, \ldots$ about O is G. Any line through O meets P_r at Q_r and P at Q. Show that

$$\frac{G}{OQ} = \frac{G_1}{OQ_1} + \frac{G_2}{OQ_2} + \ldots .$$

99. Four equal forces act *in order* along the sides AB, BC, CD, DA of a quadrilateral inscribed in a circle of which AC is a diameter. Prove that the line of action of their resultant cuts CD at a distance $(CD+DA)(\cot \frac{1}{2} BCD \backsim 1)^{-1}$ from D.

100. If $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$ are three parallelograms, and six forces, represented in magnitude and direction by A_1B_2, C_1D_2, A_2B_3, C_2D_3, A_3B_1, C_3D_1, act at a point, show that these forces are in equilibrium.

101. Two coplanar forces are represented completely by two segments A_1B_1, A_2B_2. A_1A, B_2B are drawn parallel to A_2B_1, and A_2A, B_1B parallel to A_1B_2. Prove that AB represents completely the resultant of the two forces.

102. A system of coplanar forces has moments L, M, N about three given points A, B, C in the plane, and R is the resultant. Prove that $4\Delta^2R^2 = \Sigma a^2(L-M)(L-N)$, where a, b, c, Δ are the sides and the area of the triangle ABC.

103. Forces of magnitudes μ/a, μ/b, μ/c act along the sides a, b, c of a triangle ABC taken in order; prove that the line of action of their resultant is given by the following construction. Let D, E, F be the feet of perpendiculars from A, B, C on the opposite sides. Through A draw AG parallel to EF meeting BC in G, and through B draw BH parallel to FD meeting CA in H; then GH is the required line. Show also that the magnitude of the resultant is

$$\mu\,(\Sigma a^4 - \Sigma b^2c^2)^{\frac{1}{2}}/abc.$$

104. Any number of forces in a plane are represented in direction, magnitude, and position by the straight lines OA_1, OA_2, ..., and a point Q is taken on the line of action of their resultant. Show that O lies on the line of action of the resultant of forces represented by QA_1, QA_2 ... .

105. A number of coplanar forces are in equilibrium and have as their lines of action tangents to an equiangular spiral. Prove that they will still be in equilibrium if rotated about the points of contact of these tangents through the same angle.

106. Three points A, B, C are taken in a given lamina. Forces P, Q, R act along the internal bisectors of the angles of the triangle ABC and towards the respective angular points. Determine the forces P', Q', R' which must act along the sides BC, CA, AB respectively in order that the lamina may be in equilibrium.

Result. $P' = \frac{1}{2} Q \sec \frac{1}{2} B - \frac{1}{2} R \sec \frac{1}{2} C$, &c.

107. Two smooth spheres each of radius a and weight W lie in contact in a smooth spherical bowl of radius na; prove that the pressure between them is $W/\sqrt{n^2-2n}$.

108. Forces of 1, 2, 3, 4, 5 lb. weight act round the sides of a regular pentagon in order. Find to one place of decimals the magnitude of the resultant, and prove that it acts at a distance from the centre of the pentagon of just over $3\frac{1}{2}$ times the radius of the inscribed circle.

109. Four forces, acting along the sides of a quadrilateral inscribed in a circle, are in equilibrium. Determine their ratios in terms of the sides of the quadrilateral.

Result. The magnitude of the force in any side is proportional to the length of the opposite side.

110. Points P, Q, R are taken on the sides BC, CA, AB respectively of a triangle ABC, and forces represented in direction and magnitude by AP, BQ, CR are equivalent to a couple. Show that

$$BP : PC = CQ : QA = AR : RB.$$

111. Show that a couple acting on a rigid body may be replaced by three determinate forces acting round any given triangle whose plane is perpendicular to the axis of the couple.

112. A rigid body is in equilibrium under a given couple G and three forces acting in given lines in the same plane; find the magnitudes of the forces.

113. Forces P, Q, R acting at the vertices A, B, C of a triangle respectively, in directions perpendicular to BC, are in equilibrium; prove that, if they are turned about their points of application so as to be perpendicular to AB, they are equivalent to a couple of moment $R \,.\, CF^2/BF$, where F is the foot of the perpendicular from C on AB.

Prove that $P/a = Q/b \cos C = R/c \cos B = G/(ac - c \cos B \,.\, BF)$, where G is the required moment.

114. If forces represented by AA', BB', CC', in magnitude and lines of action are equivalent to a couple, and if DG, EG, FG are parallel and equal to AA', BB', CC' respectively, prove that G is the C. G. of the triangle DEF.

ABC, $A'B'C'$ are two triangles in the same or different planes, so situated that forces represented by AA', BB', CC' are either in equilibrium or equivalent to a couple. Prove that the two triangles have the same C. G.

115. The resultant of three like parallel forces P_1, P_2, P_3 acting at A_1, A_2, A_3 meets the plane $A_1A_2A_3$ at O, and each force is inversely proportional to the distance of its point of application from O. Show that the angles subtended at O by the sides of the triangle $A_1A_2A_3$ are equal.

116. A mass of 50 lb. is suspended from a point A by two strings each of length 6 feet. The strings are attached to the same point B

of the mass and are held apart by means of a weightless rod, with smooth rings at its ends, which can slide on the strings. Prove, from considerations of the equilibrium of the weightless rod, that if the rod is horizontal there can only be equilibrium when it is half-way between A and B. If the rod is 4 feet in length, find the thrust along it. *Result.* $20\sqrt{5}$ lb. wt.

117. A uniform rod $APCQB$ of length $2a$ rests horizontally under a smooth peg P and over a smooth peg Q; C is a fixed point in the rod distant c from A; show that the pressures on P and Q are greatest when P is under A, the pegs being moveable in such a way that $PC \cdot CQ = (a-c)(a+c)^{-1}c^2$.

Let the pressures at P and Q be R and S; $W =$ weight of rod; $PC = x$, $CQ = y$. Taking moments about Q we find

$$(x+y)R = (a-c-y)W;$$

whence

$$2Rx = (a-c)W \pm \{(a-c)/(a+c)\}^{\frac{1}{2}} \{(a^2-c^2)W^2 - 4c^2R(R+W)\}^{\frac{1}{2}},$$

and $2Rc \not> (a-c)W$.

118. Two uniform beams of unequal length rest with their lower ends in contact on a smooth horizontal plane, and their upper ends against two smooth vertical parallel planes. Prove that, if their inclinations to the horizontal are 60° and 30°, the weight of one beam is three times that of the other. Prove also that the pressure on either vertical plane is one-half the geometric mean between the weights of the beams.

119. A uniform beam AB, 10 feet long and of mass 40 lb., is suspended in a horizontal position by means of two vertical ropes, one at A and the other at a point P between A and B. Find the position of the point P in order that the tensions in the ropes may be equal when a load of 20 lb. is placed on the beam 2 feet from A.

Result. 8 feet from A.

120. A uniform triangular lamina ABC in which AB is 4 in., AC is 3 in., and BC is 5 in. is suspended from the corner A, and masses P and Q, of such magnitudes that BC is horizontal, are suspended at B and C respectively. If the mass P at B is one-sixth that of the lamina, find the ratio of Q to P. *Result.* 10 : 3.

121. A uniform rod rests in equilibrium with its lower end on a smooth inclined plane and its upper end attached by a light string to a fixed point. Show that the tangents of the angles which the rod, the string, and a vertical line make with the plane are in arithmetical progression.

122. A uniform ladder rests with one end on a smooth horizontal plane and the other against a smooth vertical wall, the inclination of the ladder to the wall being 30°. If the ladder is prevented from slipping by a rope which joins a point of the ladder to the corner between the wall and the plane, so that the direction of the rope is at

right angles to that of the ladder, show that the tension in the rope is equal to one-half the weight of the ladder.

123. $ABCD$ is a square, and forces equal to 9, 2, and 1 lb. wt. act along DA, BA, and BC respectively; find the magnitude of the force along DC in order that the resultant of the four forces may pass through the centre of the square. In this case find the magnitude of the resultant. *Result.* 8 lb. wt.; 10 lb. wt.

124. $ABCD$ is a quadrilateral lamina of uniform density with sides BC and AD parallel, BC being longer than AD. Show that it can rest in a horizontal position if supported at B, C, D; and find the pressures on the supports in terms of the weight of the lamina and the ratio in which AC and BD divide one another.

125. A uniform straight rod 70 cm. long can turn freely about a point distant 31 cm. from one end. To the ends of the rod are attached the ends of a light string 98 cm. long, on which slides a smooth ring whose weight is four times that of the rod. Prove that when the system hangs in equilibrium the inclination of the rod to the vertical is $\tan^{-1} 7$.

126. A light thread of length $6a$ has equal heavy particles attached to it at the five points which divide it into six equal parts. Its ends are fixed at two points A, B in the same horizontal, and it hangs at rest with the two lowest parts of the string inclined at 30° to the horizon. Prove that

$$AB\sqrt{7} = a(\sqrt{3} + \sqrt{7} + \sqrt{21}),$$

and find the depth of the lowest particle below AB.

Result. $\frac{1}{2}a(1 + \sqrt{3} + \sqrt{3/7})$. Use a graphical method, and show that the other parts of the string are inclined at angles 60° and $\tan^{-1}(5/\sqrt{3})$ to the horizon.

127. Two uniform rods AB, BC, equal in every respect, are freely hinged at B and rest symmetrically upon a smooth parabola, whose axis is vertical and vertex uppermost. If in the position of equilibrium the rods are inclined at an angle $\pi/6$ to the vertical, show that the length of a rod is $8\sqrt{3}$ times that of the latus rectum of the parabola.

128. Two equal uniform rods AB, BC, each of weight W and length $2a$, are rigidly connected at B so as to include a right angle; a light string carrying at one end a weight W' passes over a smooth pulley and is fastened at its other end to C; A is hinged freely to a fixed point in the same horizontal plane as the pulley, and distant $4a$ from the pulley, and the system hangs in a vertical plane. If in the position of equilibrium AB is inclined to the horizon at an angle $\tan^{-1}\frac{4}{3}$, prove that $13W = 2\sqrt{10}\,W'$.

129. Two uniform heavy right cylinders, equal in all respects, are placed side by side horizontally and slung by a number of equal strings, which pass beneath them and have their ends fastened at fixed points in a horizontal line. Another heavy cylinder is placed

symmetrically on these and parallel with them. Show that, whatever its weight, it cannot separate the lower cylinders unless its radius be less than a certain fraction of their radius, this fraction depending on the inclination of the upper parts of the string to the vertical.

130. A heavy uniform beam AB, 6 feet long, is supported in a horizontal position, the supports being at C and D, where AC is 8 inches and BD is 8 inches. An exactly similar beam rests on AB with one end projecting 11 inches beyond A. If a downward vertical force is applied to the projecting end, and increased till equilibrium is broken, will the upper beam commence to turn about A, or will both beams commence to turn about C?

131. A sphere is divided by a diametral plane into two hemispheres, which are placed together and rest symmetrically on two pegs in the same horizontal, the common diametral plane being vertical. Prove that the least distance of the pegs apart in order that the hemispheres may not separate is to the diameter of the sphere as 3 to $\sqrt{73}$.

132. Two points AB of a heavy body are constrained to move on the parabola $y^2 = 4ax$ whose axis is vertical and whose vertex is lowest. The length of AB is $12a$. The C. G. (G) of the body is above AB and the perpendicular from G on AB bisects AB and is of length $7a$. Show that the only positions of stable equilibrium are those in which AB makes an angle of $30°$ with the vertical.

133. A bent lever, with the fulcrum at the bend, is worked from one end so as to exert pressure by the other end in a given direction. Find the direction in which force must be applied in order that a given pressure may be exerted with the least strain on the fulcrum.

134. Two equal uniform rods AB, BC, freely jointed at B, are freely suspended from a point A so that the end C rests without friction against a horizontal plane; find the position in which they will rest.

135. A uniform rod $PQRS$, of length $3a$ and weight W, is bent into three equal parts at Q and R, so that each part is perpendicular to each of the other two. If the rod is suspended by a string attached to P, show that the action at R consists of a force $\frac{2}{3}W$ and a couple $5Wa/3\sqrt{14}$.

136. Two equal cylinders lie along the bottom of a groove with plane faces, of which the base is horizontal and the sides inclined at an angle α to the horizontal, the cylinders just fitting it so that each touches the bottom and one side of the groove while they touch each other. What condition must be satisfied in order that a cylinder of equal radius placed on the top of them may not crush them out, there being no friction? Does the result depend on whether the mass of the third cylinder is great or small? Does it make a difference if it is rough?

137. A cone, the semi-vertical angle of which is $\tan^{-1}(3/8)$, rests with its vertex against a smooth vertical wall, a point in the base being connected with a point in the wall by a string which, when there is equilibrium, is parallel to the axis of the cone. Prove that this axis is inclined at an angle of 45° to the vertical.

138. A wheel of radius a is free to roll on a horizontal plane sufficiently rough to prevent sliding: it is pulled by a cord coiled round its axle, of radius c, which leaves the axle on its lower side at an inclination θ to the horizontal: find the condition on which it depends whether the wheel will roll forward or backward on account of the pull.

139. The smooth curve

$$(x^2+3y^2-12by)^2+4x^2(y+2b)^2=16a^2x^2$$

is in a vertical plane with the axis of x horizontal. From the point $(0, 4b)$ a rod of length $2a$ is suspended by a string of the same length. Show that the rod will rest on the curve in any position.

The string and rod being inclined at angles θ and ϕ to the vertical, prove that the abscissa of the intersection of $y = x\cot\theta + 4b$ with the normal at $\{2a(\sin\phi - \sin\theta),\ 4b - 2a(\cos\phi + \cos\theta)\}$ is the same as that of the middle point of the rod.

140. A cylindrical lamp-shade 2 in. in diameter, standing upright on a horizontal table, has two balls $1\frac{1}{3}$ in. in diameter dropped into it. Prove that the lamp-shade will overturn if its weight is less than $\frac{2}{3}$ the weight of the ball which is dropped in last, the balls being perfectly smooth.

141. A rigid body, of weight W, has two points A and B fixed and is free to move about the line AB; if AB is inclined at an angle α to the vertical and the distance of the centre of gravity G from AB is a, show that the couple that must be applied to the body to keep it in the position in which the plane AGB is inclined at an angle β to the vertical plane through AB is $Wa\sin\alpha\sin\beta$.

142. Coplanar forces P_1, P_2 acting at A, P_3, P_4 at B, and P_5, P_6 at C are in equilibrium. If the magnitudes of P_1, P_2, P_3 are given and the directions of all the six forces, obtain a graphic construction, by means of the degenerate funicular polygon ABC or otherwise, for the magnitudes of the forces P_4, P_5, P_6.

143. Two equal uniform rods AC, BC are freely jointed at C, and the ends A and B are free to slide on a smooth parabolic wire, whose axis is vertical and vertex upwards; prove that, in a position of equilibrium in which AB is not horizontal, it must cut the axis in a point which divides the distance between the vertex and the focus in the ratio of three to one.

144. To two consecutive corners A, B of a uniform square lamina are attached small smooth rings, which slide on a fixed wire in the form of a parabola of latus rectum p whose axis is vertical and

vertex upwards. If the lamina rests in equilibrium with the side AB above the centre and inclined at an angle $\cos^{-1}(5/6)$ to the horizontal, show that $25\ AB = 24\,p$. Is the equilibrium stable or unstable?

145. A heavy beam whose centre of mass divides its length in the ratio 1 : 2 is in equilibrium with its ends pressing against two smooth straight grooves in the same vertical plane inclined to the horizon at equal angles and in opposite directions. Find the inclination of the beam to the horizon. *Result.* $\tan^{-1}(\frac{1}{3}\cot\alpha)$.

146. Two heavy particles can slide on two equal smooth circular wires in a vertical plane. A light string in a state of tension passes through two smooth rings, fixed, one on each circle vertically above its centre, and has its extremities joined to the particles. Prove that in the position of equilibrium the distance of each particle from the ring on the circumference of its circle is inversely proportional to its mass.

147. A wire bent into the form of an ellipse is fixed with its minor axis vertical. Find the positions of equilibrium of a heavy bead strung on the wire, under the action of an attractive force in the centre which varies as the distance.

148. A rod passes through a smooth ring fixed in a vertical plane A, and has one end hinged to a point in the axis of a half cylinder, resting with its curved surface on a perfectly rough inclined plane which is perpendicular to the plane A. Find the condition that the system may be in equilibrium when the rod is horizontal, the axis of the cylinder is perpendicular to the plane A, and the rectangular base of the half cylinder is parallel to the inclined plane. Determine also whether the equilibrium is stable.

149. A heavy sphere of radius r rests on a smooth inclined plane of angle α and is prevented from sliding down by a heavy cone of the same material, and of vertical angle 2α, which rests in contact with the sphere and with its vertex moveable about a fixed pivot at a point on the plane lower than the sphere. Show that if

$$64\,r^4 \sin\alpha = 3\,a^4 \cos^3 2\,\alpha,$$

the positions (if any) of equilibrium are given by the equation

$$\cos^2\theta - 2\cos\theta\cos 2\,\alpha + \cos^2 2\,\alpha \cot^2 2\alpha = 0,$$

θ being the inclination of the axis of the cone to the horizon, and a the length of a slant side.

150. $ABCD$ is a square board of side a, and a smooth peg is at a distance c ($> a$) from a smooth vertical wall. The board rests with its plane vertical, A pressing against the wall and CD against the peg. Prove that AD makes an angle α with the horizon given by

$$2\,c/a = 2\cos^3\alpha + 3\cos\alpha\sin^2\alpha + \sin^3\alpha.$$

If AD makes an angle θ with the vertical and the peg is distant b from D, then $\bar{y}$ = height of C.G. above peg

$$= \tfrac{1}{2}\{a\cos\theta - b\sin\theta + (a-b)\sin\theta\},$$

where $a\sin\theta + b\cos\theta$ = constant. We have

$$\delta\bar{y} = 0,\ \delta(a\sin\theta + b\cos\theta) = 0.$$

Eliminate $\delta\theta$ and δb.

151. A uniform triangular plate, with sides a, b, c and circum-radius R, lies without friction in a spherical cavity of radius r; show that the inclination θ of its plane to the horizontal is given by $(r^2 - R^2)\tan^2\theta = R^2 - \frac{1}{2}(a^2+b^2+c^2)$. How must the plate be loaded so that it may lie horizontally in the cavity?

152. $ABCDE$ is a pentagon. A parallel to BD through C meets ED in V; a parallel to CE through D meets BC in U; L is any point in UV; parallels to AB, AE through L meet BC, ED in M and N respectively. Prove that a system of forces acting along the sides of the pentagon $ABCDE$ represented in magnitude and direction by the sides of the pentagon $LMCDN$ will be in equilibrium.

153. Find the magnitude of the horizontal force which will keep a body of weight W at rest on a smooth plane, which is inclined to the horizontal at an angle α.

Show that, if the body were kept at rest on the plane by a tight string, attached to a point above the plane and making an angle β with the upper part of the plane, the reaction between the body and the plane would be $W\cos(\alpha+\beta)\sec\beta$.

154. A uniform circular sector $OABC$ of weight w (OA, OC being the bounding radii and B the middle point of the arc AC) can turn freely about a horizontal axis through O perpendicular to the plane of the sector, and therefore hangs initially with OB vertical. A weight W is now attached to A. Show that in the new position of equilibrium the line OB produced will meet a line through the initial position of B parallel to the initial direction of OC at a point whose distance from the initial position of B is directly proportional to W.

Find the distance when $W = w$. *Result.* $\frac{3}{2}OA \cdot \alpha \operatorname{cosec}\alpha$.

155. A homogeneous spheroid of weight $32W$ is divided into an infinite number of separate thin slices by planes through its axis, and is held together by a light string lying in a shallow groove in the equatorial plane. If the spheroid is at rest on a horizontal plane, which it touches at an extremity of its axis, show that the tension of the string is not less than $3aW/c$, where $2c$ and $2a$ are the lengths of the polar and equatorial diameters of the spheroid.

156. A uniform heavy spheroid is divided into six equal parts by planes through its axis of figure. The parts are then glued together and the spheroid is suspended from a fixed point by six equal light strings, whose lower ends are attached to points on the equator of the spheroid such that each string lies in a plane through the axis of figure midway between two of the former planes. Show that, if the

temperature is now raised until the glue melts, the parts will separate unless the tangent of the inclination to the vertical of the upper part of a string exceeds $175a/288c$, where a and c are the equatorial and polar semi-axes of the spheroid.

Let C be the highest, and C' the lowest point. Prove that the distance of the C.G. of any of the parts from CC' is $9a/16$. If $W =$ wt. of each part, α the inclination of the free part of each string to the vertex, we see (i), by taking moments about C, that there will be no separation at C' if

$$9Wa/16 > W \sec\alpha\left(\sqrt{c^2\sin^2\alpha + a^2\cos^2\alpha} - c\sin\alpha\right)$$

or $288c\tan\alpha > 175a$; (ii), by taking moments about C', that there will be no separation at C.

157. To the ends of a uniform rod small rings are attached, which slide on a smooth wire in the form of a parabola in a vertical plane, the axis of the parabola being horizontal. The rod is kept in equilibrium by a couple in the plane of the wire. Prove that the magnitude of this couple for various positions of the rod varies inversely as the square of the difference of the distances of the ends of the rod from the axis of the parabola.

158. A torsion-balance consists of a heavy uniform bar, of length $2a$ and weight W, suspended from its ends by two equal strings, each of length l, attached to points at a distance $2a$ apart in the same horizontal line. Find the couple (whose plane is horizontal) required to maintain the bar in equilibrium, when deflected through an angle θ about the vertical through the centre of the rod.

159. A solid parallelepiped of uniform density rests with three of its faces, which meet in a point, on three smooth pegs A, B, C, in the same horizontal plane. If A', B', C', the middle points of the three faces, form a triangle equal and similar to ABC, show that the solid will be in equilibrium when so placed that A', B', C' coincide with A, B, C. Hence prove that if a solid ellipsoid were substituted for the parallelepiped, it would be in equilibrium when so placed that A, B, C were the ends of conjugate diameters.

160. A heavy homogeneous ellipsoid is supported in equilibrium by three smooth pegs, the points of support being the ends of three conjugate semi-diameters. Prove that the pegs must be in the same horizontal plane and that the pressures on them are to one another as the areas of the corresponding conjugate central sections.

Let the pegs be P, Q, R; their co-ordinates referred to the principal axes of the ellipsoid (al_1, bm_1, cn_1), $(al_2, \ldots, \ldots)$, $(al_3, \ldots, \ldots)$; p_1, p_2, p_3 the perpendiculars from the centre on the tangent planes at P, Q, R; R_1, R_2, R_3 the reactions. Find the components of the reactions parallel to the axes, and, by taking moments about the axes, prove that $R_1p_1 = R_2p_2 = R_3p_3$. By resolving parallel to the axes, prove that the weight acts perpendicularly to the plane

$$\Sigma x(l_1 + l_2 + l_3)/a = 1.$$

161. The co-ordinate planes are smooth rigid boundaries, and the direction cosines of the vertical are l, m, n. Prove that if a rigid body rests in contact with the planes respectively at

$$(0, y_1, z_1), (x_2, 0, z_2), (x_3, y_3, 0),$$

then $$mn(x_2-x_3)+nl(y_3-y_1)+lm(z_1-z_2)=0;$$

and find the equations of the vertical through the centre of gravity.

If the body is a homogeneous ellipsoid of semi-axes a, b, c, prove that the vertical height of the centre above the origin is

$$\{a^2+b^2+c^2-(mz_2-ny_3)^2-(nx_3-lz_1)^2-(ly_1-mx_2)^2\}^{\frac{1}{2}}.$$

Result. $m(z-z_2)=n(y-y_3)$, $n(x-x_3)=l(z-z_1)$.

The relation to be proved is the condition for a single resultant ($\Sigma LX=0$).

162. If forces acting along the given non-coplanar lines OA, OB, OC, OD are in equilibrium, show that their ratios are fixed, and determine them in terms of the cosines of the angles between the lines.

Forces P, Q, R act along the sides BC, CA, AB of a triangle, and D is any point not in the plane of the triangle. Show that the moment of the forces about any line through D cannot exceed G, where

$$G^2+\begin{vmatrix} 1, & \cos\nu, & \cos\mu, & Pb'c'/a \\ \cos\nu, & 1, & \cos\lambda, & Qc'a'/b \\ \cos\mu, & \cos\lambda, & 1, & Ra'b'/c \\ \dfrac{Pb'c'}{a}, & \dfrac{Qc'a'}{b}, & \dfrac{Ra'b'}{c}, & 0 \end{vmatrix}=0,$$

where λ, μ, ν are the angles BDC, CDA, ADB, and a, b, c, a', b', c' are the lengths BC, CA, AB, DA, DB, DC.

Take DA, DB, DC as oblique axes of co-ordinates. The force P along BC is equivalent to a parallel force P through D and a couple of moment $G_1\equiv Pb'c'\sin BDC/a$ in the plane BDC. If the axis of G_1 makes an angle α with DA and has l_1, m_1, n_1 for its direction ratios, then $l_1+m_1\cos\nu+n_1\cos\mu=\cos\alpha$,

$$l_1\cos\nu+m_1+n_1\cos\lambda=0=l_1\cos\mu+m_1\cos\lambda+n_1.$$

So the forces Q, R along CA and AB are equivalent to forces Q, R at D and couples of moments G_2, G_3. The forces P, Q, R at D have no moment about a line passing through D. If the direction ratios of such a line are l', m', n', the moment of the (original) forces about it is $G\equiv G_1 l'\cos\alpha+G_2 m'\cos\beta+G_3 n'\cos\gamma$, where β, γ are the angles which the axes of G_2, G_3 make with DB, DC respectively. G is to be a maximum when l', m', n' vary subject to the relation

$$1=l'^2+m'^2+n'^2+2m'n'\cos\lambda+2n'l'\cos\mu+2l'm'\cos\nu.$$

If $M^2=1-\Sigma\cos^2\lambda+2\cos\lambda\cos\mu\cos\nu$, prove that

$$l'+m'\cos\nu+n'\cos\mu-Pb'c'M/aG=0, \ldots,$$

and $Pb'c'Ml'/a+\ldots+\ldots-G=0$. Eliminate l', m', n' and simplify.

163. A piece of uniform wire is formed into a triangle ABC, and is suspended from a fixed point O by three light strings attached to the angular points. If T_1, T_2, T_3 are the tensions of these strings, prove that

$$\frac{T_1}{(AB+CA)\,OA} = \frac{T_2}{(BC+AB)\,OB} = \frac{T_3}{(CA+BC)\,OC}.$$

If G is the C.G. of the wire and CG meets AB in D, then the resultant of T_1 and T_2 acts along DO, and $T_1 . DA/OA = T_2 . DB/OB$. The ratio $DA : DB$ may be found from the trilinear equation of CGD.

164. Three fixed planes are mutually at right angles and have different inclinations to the vertical. A smooth solid rests in equilibrium in the upmost octant between them with one point in contact with each plane. Two of the three points of contact are given points on two of the planes. Prove that the third point of contact lies on a fixed line of greatest slope of the third plane, and that the C.G. of the solid lies in a fixed vertical plane parallel to this line of greatest slope.

165. In that system of pulleys where each string is attached to a bar from which the weight is hung and all the strings are parallel, the radius of each of the n pulleys is 4 inches, and their weights are negligible in comparison with that of the bar; find the position of the centre of gravity of the latter, if it rests in equilibrium when both ends of the string round the lowest pulley are attached to it, the strings where not in contact with pulleys being all vertical.

166. A uniform beam 4 feet long and weighing $9\frac{1}{4}$ lb. is supported by three vertical cords passing over three pulleys in that system in which each cord is fastened to the weight. Each pulley weighs 9 oz. and the cords are fastened to the middle point of the beam and to points 6 inches on each side of the middle point. Prove that, if a weight of 21 oz. is attached to the end of the beam nearest to the string whose tension is greatest, the beam can rest in equilibrium in a horizontal position, provided the power is properly chosen; and find the value of the power.

167. If the least force which can maintain in equilibrium a weight Q hanging freely over a single pulley is, owing to the rigidity of the cord and to friction, equal to the weight $aQ+p$, show that, in the system of pulleys in which each string is fastened to the weight, the greatest power P insufficient to raise a weight W is given by

$$a(W+nw) = \{(a+1)^n - 1\}\,(aP+w+p),$$

where w is the weight of each pulley, and n the number of pulleys.

168. A uniform ladder AB, weighing 50 lb., rests against a smooth vertical wall at A and on a rough horizontal plane at B. Find the coefficient of friction between the ladder and the horizontal plane if it is about to slip when inclined at 45° to the vertical. Find also what force must be exerted at B, applied by means of a string attached to B and inclined at 30° to the upward-drawn vertical, in order that the ladder may begin to move towards the wall.

Result. $\frac{1}{2}$; $200(2-\sqrt{3})$ lb. wt.

169. The bases of a frustum of a solid homogeneous cone are 2 and 3 feet in radius, the thickness is 8 feet; the frustum is placed with its smaller face on a rough plane whose inclination to the horizon is gradually increased. If the coefficient of friction is $\frac{1}{2}$, will the frustum slide or topple over?

170. A ladder rests with one extremity against a vertical wall and the other on the ground. When the inclination of the ladder to the vertical is $60°$ it is about to slip down; assuming that the ground and wall are equally rough, find their coefficient of friction with the ladder.

If a rope is attached to the upper end of the ladder and passed over a pulley vertically above that end, and a gradually increasing weight attached to the extremity of the rope, show that when equilibrium is on the point of being broken there will be no pressure between the ladder and the wall.

171. A heavy uniform beam rests with one end on horizontal ground, and the other end on a rough plane inclined at an angle α to the horizon. If ϕ is the angle of friction for the contact of the beam with both the plane and the ground, show that the inclination β of the beam to the horizon when it is on the point of slipping is given by the equation

$$2 \tan \beta = \cot \phi - \cot (\alpha - \phi).$$

If $\alpha = 60°$ and $\beta = 30°$, find the value of ϕ.

172. A uniform heavy rod 1 foot long, one end of which is rough and the other smooth, rests within a circular hoop in a vertical plane whose radius is 10 inches. If the rod is in a position of limiting equilibrium when its rough end is at the lowest point of the hoop, show that the coefficient of friction is $\frac{24}{43}$.

173. A rough peg is fixed at a perpendicular distance $3a$ from a plane of equal roughness inclined at an angle of $30°$ to the horizon; a uniform rod of length $4a$ rests against the peg with its lower end supported by the plane; if, when the rod is inclined at an angle of $30°$ to the horizon, its lower end is on the point of slipping down the plane, find the coefficient of friction.

174. Two particles of masses m_1 and m_2 are held at rest each on a horizontal table and connected by a string which passes over small smooth pulleys at the edges of the tables and is perpendicular to these edges. A smooth pulley to which a weight is attached, and whose diameter is equal to the distance between the former pulleys, hangs between the tables in the loop of the string. The combined mass of the hanging pulley and attached weight is M, the coefficients of friction between m_1 and m_2 and the tables on which they are held are μ_1 and μ_2 respectively. Show that, if $2m_1\mu_1$ and $2m_2\mu_2$ are each of them not less than M, equilibrium will not be disturbed, if the particles are released; and that, if $m_1\mu_1 < m_2\mu_2$ and $M > 2m_1\mu_1$, m_2 will not move, if

$$2m_1(M - 2m_2\mu_2) \ngtr M(m_2\mu_2 - m_1\mu_1).$$

175. From a uniform sphere a portion equal to $\frac{5}{32}$ of the whole is cut off by a plane. This segment rests with its plane face on a rough inclined plane and is on the point of sliding down. A light string attached to the vertex of the segment is then pulled in a direction parallel to and up the plane, and the tension is gradually increased until the body begins to move. Prove that it will tilt or slide according as $33\mu \gtrless 20\sqrt{3}$, where μ is the coefficient of friction.

Prove that the plane face bisects the perpendicular radius (a) of the sphere, and that the C. G. of the segment is distant $27a/40$ from the centre of the sphere. If $\mu = \tan\alpha$, $T_1 =$ tension for just tilting, $T_2 =$ tension for just sliding, prove, by taking moments, that

$$T_1 = W(\sqrt{3}\cos\alpha + 7\sin\alpha/20),\ T_2 = 2W\sin\alpha.$$

The segment will slide if $T_1 > T_2$.

176. A symmetrical beam AB of weight W rests at an angle of 45° to the horizon with its lower end A on a horizontal floor and its upper end B against a vertical wall. If the floor and wall are equally rough, find the smallest value of the coefficient of friction μ which is consistent with the equilibrium.

If μ lies between this value and $\frac{1}{2}$, show that the vertical pressure at A is not less than $\dfrac{\mu+2}{2(\mu+1)}W$, or greater than $\dfrac{W}{2(1-\mu)}$.

177. The two legs of a pair of steps are of equal lengths and of weights W, W' ($W > W'$), and the steps will just stand on a rough ground when the legs contain an angle 2α; assuming that the coefficients of friction for the two feet are the same, prove that this coefficient is equal to $(W + W')\tan\alpha/(W + 3W')$.

178. Two circular cylinders have radii a, $a\sqrt{3}$ respectively, and have their axes parallel, in the same horizontal plane, and at a distance $4a$ apart. Prove that a heavy rod of length $2a$ cannot rest horizontally with an end on each of them unless the coefficients of friction between it and the cylinders are both equal to, or at least one greater than, tan 15°.

179. AB, AC are two uniform heavy rough rods, which rest in equilibrium with their ends B and C on a rough horizontal plane, whilst the ends A rest against one another, the rods being so cut that the surfaces in contact at A are small vertical planes. If the coefficients of friction (i) between each rod and the horizontal plane, (ii) between the two rods, are each equal to $\tan\lambda$, and if the equilibrium is limiting at each point of contact, prove that the weights of the rods must be proportional to the lengths of their projections on the plane, and that $\cot ABC = \cos ACB = \sin 2\lambda$, where ABC is the greater of the two angles ABC, ACB.

180. A light bar AB rests horizontally with the end A in contact with a rough vertical wall, being supported also by a string OC connecting a point O of the rod to a point C of the wall, vertically

above A. Find, graphically, from what part of the length of the bar a weight can be suspended without causing the end A to slip.

Find the coefficient of friction at A, if slipping takes place when the weight is at a distance of 40 in. from A, the lengths AO and AC being 30 in. and 25 in. respectively.

181. A uniform heavy wire, which is in the form of an ellipse of eccentricity $\cos\alpha$, is hung over a small rough peg. Prove that, if the wire can be in equilibrium with any point in contact with the peg, the coefficient of friction must not be less than $\frac{1}{2}\cos^2\alpha \operatorname{cosec}\alpha$.

182. A rough disk of radius a and weight W standing vertically on a rough horizontal plane has resting against it in the same vertical plane a uniform rod of length $2l$ and weight w, the lower end of which can turn freely about a hinge in the horizontal plane. If equilibrium is limiting at both points of contact simultaneously, show that

$$lw(\tan\epsilon\cot\lambda - 1) = aW\cot\epsilon\sec 2\epsilon,$$

where ϵ, λ are the angles of friction between the rod and disk and disk and plane respectively.

183. One end of a thin uniform rod, of length $2a\cot\alpha$ and weight W, is freely hinged to a point on a fixed plane, which is inclined at an angle 2α to the horizontal. A uniform sphere of radius a and weight W rests on the plane and has the rod for a tangent at its highest point. The vertical plane through the rod and the centre of the sphere cuts the fixed plane in a line of greatest slope. If the system is in equilibrium and the friction is limiting at each point of contact, determine the coefficients of friction.

Result. $\tan\alpha$ and $2\tan\alpha$.

184. A rough horizontal bowl is fixed with its diametral plane horizontal, and a heavy rod rests over the rim with its lower end in contact with the inside of the bowl. If α and β are the greatest and least angles which the rod can make with the horizon, prove that the coefficient of friction is

$$(\cos\alpha\cos 2\beta - \cos\beta\cos 2\alpha)/(\sin 2\alpha\cos\beta + \sin 2\beta\cos\alpha).$$

185. A rough stick rests on the rim of a flower-pot, whose shape is that of a truncated cone of semivertical angle α. Its lower end is in contact with a slant side, the vertical plane through the stick passing through the axis of the cone. Prove that if the stick is inclined to the horizontal at an angle β, when it is on the point of slipping down at each point of contact, then its length $2l$ is given by

$$l\cos\beta\cos(\beta-\alpha)\cos(\beta-\alpha-2\lambda) = c\cos\alpha\cos\lambda\cos(\alpha+\lambda),$$

where λ is the angle of friction and c the diameter of the flower-pot at the top.

186. A rough wire of length $3a$ is bent into the form of an equilateral triangle. Each side of the triangle passes through a weightless ring, and the rings are connected together by an endless smooth tight string of length $3l$ which passes through each of them.

If each of the rings is in limiting equilibrium when they are at the vertices of another equilateral triangle, find the coefficient of friction between a ring and the wire.

187. Equal masses are placed on a rough horizontal plane at consecutive corners A, B, C of a regular dodecagon $ABCD$......, and the mass at B is connected to those at A and C by light strings which are just tight. A gradually increasing force acts on C in the direction CD. Show that the three masses will begin to move simultaneously, and find the directions of their motions.

188. A rigid body of weight W, pierced with a cylindrical cavity of radius b, can turn about a fixed horizontal axle which just fits the cavity. The centre of gravity of the body is at a distance a from the axis of the axle, in a horizontal line perpendicular to this axis, and the body is supported by a vertical force T, applied at a point distant c from the axis, in the line drawn through the centre of gravity at right angles to the axis. Prove that, if there is friction between the axle and the cavity at points on a single generator of the cavity, and ϵ is the angle of friction, the least value of the force T is $W(a-b\sin\epsilon)/(c-b\sin\epsilon)$, the lengths b, a, c being in ascending order of magnitude.

189. A uniform solid cylinder of weight W rests in the angle between two equally rough planes inclined to the vertical at the same angle α on opposite sides, the line of intersection of the planes being horizontal. A particle of weight W is attached to the cylinder at a point on the level of its axis. If there is limiting friction at both points of contact and μ is the coefficient of friction between the cylinder and either plane, show that $\mu = \tan\frac{1}{2}\alpha$ and $\alpha < 60^\circ$.

190. A uniform rod touches a smooth fixed hemisphere, whose radius is a quarter of the length of the rod, and rests in a vertical plane through the centre of the hemisphere, with its lower end on a rough horizontal plane through the base of the hemisphere. Prove that, if λ is the angle of friction between the rod and the plane, the rod will rest in limiting equilibrium inclined at an angle $\cot^{-1}(1+\sqrt{2\cot\lambda})$ to the horizontal, provided $2\cot\lambda$ is not greater than 9.

191. Two equal uniform rods freely hinged together at their upper ends rest symmetrically upon a rough fixed sphere with the hinge vertically above the centre of the sphere. Given that the angles between the rods in the two positions of limiting equilibrium are 60° and 120° respectively, find the coefficient of friction, and the ratio of the length of a rod to the radius of the sphere.

192. A particle placed on a rough inclined plane is attached to a point of the plane by means of a light inextensible string. The particle is placed so that the string is taut, and the plane is slowly tilted about a horizontal axis which makes an angle α with the string, the direction of tilting being such that the point of attachment is raised higher than the particle. Find in terms of α and of μ, the

coefficient of limiting friction between the particle and the plane, the angle through which the plane can be tilted before the particle slips.

193. A solid uniform cone of weight W and vertical angle 2β has three small knobs symmetrically placed on the boundary of its base, and rests with these in contact with a rough plane inclined to the horizon at an angle α. The line joining two knobs is horizontal and above the third, which is prevented from sliding. Prove that, if the cone be rotated on the plane, it will slip when it has turned through the smallest angle θ which satisfies the equation

$$\sqrt{3}\sin\theta + \tfrac{1}{2}\mu\cot\beta\cos\theta = 2\mu\cot\alpha,$$

where μ is the coefficient of friction, provided that

$$2\tan\beta > \tan\alpha > \frac{2\mu}{\sqrt{(3+\frac{1}{4}\mu^2\cot^2\beta)}}.$$

194. A uniform heavy rod has a string attached to its ends. The string passes through a small smooth ring which is fixed to a rough vertical wall, and one end of the rod rests against the wall, the rod and the string being in a plane perpendicular to the wall. Prove that, if the length of the rod is μ times the length of the string, where μ is the coefficient of friction, the rod cannot rest in equilibrium with the end that is against the wall higher than the other end.

195. A uniform rod of weight W has its upper end against a vertical plane and its lower end against a horizontal plane, and is inclined to the vertical at an angle α ($< 45°$). The vertical plane through the rod is at right angles to the line of intersection of the former planes and meets it at A. B is the foot of the perpendicular on the rod from A. The rod is kept in equilibrium by a light string joining A to B. The coefficient of friction at each end of the rod is $\tan\lambda$ ($\lambda < \frac{1}{2}\alpha$). Show that the tension of the string is not less than

$$\tfrac{1}{2}W\sin(\alpha - 2\lambda)\sec(2\alpha - \lambda)\sec\lambda$$

and not greater than

$$\tfrac{1}{2}W\sin(\alpha + 2\lambda)\sec(2\alpha + \lambda)\sec\lambda.$$

196. A uniform rod BC, of weight W, has at its ends weightless rings which slide on two fixed rough wires AB and AC, which are at right angles. The coefficient of friction between the wire and the ring is in each case $\frac{1}{2}$. The middle point of the rod is attached to A by a weightless string. If the rod is in limiting equilibrium when the wire BA is vertical, with the end A uppermost, and when the angle ABC is $45°$, show that only two sets of values for the reactions at the rings are possible, and that the tension of the string is either

$$7\sqrt{2}W/8 \quad \text{or} \quad \sqrt{2}W/8.$$

197. A rough wire in the form of a parabola of latus rectum $4a$, with its axis vertical and vertex upwards, rotates with uniform angular velocity $\sqrt{g/2a}$ about its axis. A small ring of mass m can

slide on the wire and is attached to the focus by an elastic string of natural length a, which can be stretched to double its length by a tension equal to $\frac{2}{3}mg$. Prove that the ring will remain in equilibrium at any point of the wire the distance of which from the axis is less than $a\mu$, where μ is the coefficient of friction between the ring and the wire.

If we apply to the ring an outward force $mgy/2a$ along the ordinate y, the problem is reduced to a statical one.

198. Two planes at right angles to one another slope upwards in opposite directions from their line of intersection, which is horizontal. The steeper plane is rough and the other plane smooth. A uniform plank rests in a horizontal position perpendicular to the intersection of the planes with an end on each plane, and is in limiting equilibrium. When a weight equal to that of the plank is placed upon it close to that end which rests on the smooth plane the equilibrium is again limiting. Determine the angle of friction and the inclination of the rough plane to the horizontal.

Result. If α is the inclination, $2\alpha = \lambda + \frac{1}{2}\pi$ and $3\tan^2\alpha = 5$.

199. AB and BC are two uniform ladders of equal lengths, and weights P and Q, respectively; they are freely jointed together at B by a smooth axis; the extremity A is fixed by a smooth horizontal axis, while C moves along a rough horizontal plane passing through A; the plane ABC is vertical, and μ is the coefficient of friction between C and the ground; find the greatest value of the angle ABC, and the corresponding magnitude and direction of the mutual pressure at B.

200. A cylinder is laid on a rough horizontal plane and is in contact with a rough vertical wall. A string coiled round the cylinder at right angles to its axis, after leaving the surface at a point below the axis, passes over a smooth pulley which is fixed above the axis at a distance from the wall greater than the diameter of the cylinder. To the end of the string a weight is attached which is gradually increased until equilibrium is broken. Determine the manner in which this occurs.

If the weight of the cylinder is 53 lb., the inclination of the string to the horizon $\cos^{-1} 3/5$, and the coefficient of friction at each point of contact $2/5$, find the value of the suspended weight which is just sufficient to break the equilibrium.

Result. 17 lb. approx. See § 158, Ex. 11.

201. A small heavy ring is moveable on a rough wire in the form of a parabola whose plane is vertical and axis horizontal. Two points P, Q on the upper part of the wire subtend at the focus an angle equal to four times the angle of friction. Prove that the tangential force that will just sustain the ring at the lower of the two points is equal to the force that will just drag the ring along the wire at the higher point.

202. A uniform rod rests in equilibrium with its extremities on a rough elliptic hoop fixed in a vertical plane; show that if the tangents

to the hoop at the extremities of the rod contain a right angle, the angle between the vertical and the connector of the centres of the rod and hoop is twice the angle of friction.

203. A hollow circular cylinder, open at the top, rests with its base on a rough horizontal plane. Its height $2a$ is equal to the diameter of the base. Inside it is a uniform rod of the same weight as the cylinder and of length $4a$, resting symmetrically with one end at a point in the perimeter of the base and leaning against the upper edge of the cylinder. What weight must be placed at the top of the rod that the whole may be just on the point of toppling over?

When this weight has been attached, find the coefficient of friction of the base with the plane if, when a horizontal force is applied to the same end, the rod is on the point of being lifted from the rim and the cylinder is on the point of slipping at the same time.

204. Show that a plank can be balanced horizontally across a fixed perfectly rough horizontal bar of circular section so as not to be upset by a slight disturbance, provided the thickness t of the plank is less than the diameter d of the bar; and that it can then be tilted up to any angle less than the angle given by the formula $d\alpha = t \tan \alpha$ without falling off.

205. A homogeneous elliptic cylinder rests in contact with rough inclined planes whose line of intersection is horizontal and parallel to the axis of the cylinder. Show that when the cylinder is about to slip down the plane whose inclination to the horizon is i', the major axis of the principal section makes an angle θ with the vertical where

$$\frac{\sin(i-\epsilon)}{\sin(i'-\epsilon')} = \left\{\frac{1-e^2\sin^2(\theta+i')}{1-e^2\sin^2(\theta+i)}\right\}^{\frac{1}{2}} \frac{e^2\sin(2i+2\theta-\epsilon)-(2-e^2)\sin\epsilon}{-e^2\sin(2i'+2\theta-\epsilon')+(2-e^2)\sin\epsilon'},$$

i and i' being the inclinations of the planes, measured in the same sense, to the horizon; ϵ and ϵ' the angles of friction, and e the eccentricity of the principal elliptic section.

206. A uniform bar is supported by two rough pegs, whose distance apart is equal to c and makes an angle α with the horizontal, by passing over one of them and under the other; show that it cannot rest in this way unless its length exceeds $2c\ (\mu_2+\tan\alpha)/(\mu_2+\mu_1)$, where μ_1 and μ_2 are the coefficients of friction at the pegs respectively nearer to and further from the C.G.

207. $ABCD$ is a tetrahedron of which the edges DA, DB, DC are mutually at right angles. Its face ABC rests on a plane inclined at an angle i to the horizon and sufficiently rough to prevent sliding. Prove that the tetrahedron will not topple over in any position on the plane if $\tan i < p/n$, where p is the least of the quantities $\cos(B-C)$, $\cos(C-A)$, $\cos(A-B)$, and n is the ratio of the height of the tetrahedron to the diameter of the circle ABC.

208. A, B, C are the vertices of an equilateral triangle inscribed in the circle $x^2+y^2=a^2$, the point C being at $(-a, 0, 0)$ and the plane of xy

horizontal. A uniform triangular lamina is supported in a horizontal position by three equal similar legs, which are attached to its vertices and which rest upon the plane of xy at A, B, C, the coefficients of friction μ_1, μ_2, μ_3 being all different. Through the action of horizontal forces the triangle is on the point of turning about the axis $x = r\cos\alpha$, $y = r\sin\alpha$. Obtain the condition that the resultant of the three frictions is perpendicular to the line $\theta = \alpha$, and find the relation between r and α when this condition is satisfied, $\mu_1 = \mu_2$ and μ_3 is negligible.

209. The coefficients of friction between a sphere and three rods of different lengths are μ_1, μ_2, μ_3. The rods are joined at their ends to form a triangle, which is held fixed and horizontal. The sphere is then placed above the triangle and sustained by it. Prove that if a couple is applied to the sphere about a vertical axis, and if it just causes the points of contact to move simultaneously along the respective rods, then must $\mu_1 = \mu_2 = \mu_3$.

210. A heavy uniform plank of length $2l$ rests symmetrically across a rough beam whose section is a semicircle of radius a, and has a weight W at each end; it is then carefully tilted without slipping through an angle θ. Show that it will rest in this position if a force P is applied at the lower end perpendicular to the plank, provided that $(l - a\theta)\sin\theta$ is not greater than $\mu l\cos\theta$, where μ is the coefficient of friction, and find the magnitude of P.

Show also that with a weight W_1 at the upper and a weight W_2 at the lower end the plank will rest at an angle $\epsilon\,(= \tan^{-1}\mu)$ to the horizon, provided that $W_1(l + a\epsilon) + wa\epsilon = W_2(l - a\epsilon)$, where w is the weight of the plank.

211. A board of uniform thickness and density can rest on a horizontal plane on three studs attached to one of its faces at the corners of an equilateral triangle ABC of side a, and its centre of gravity is vertically over the centroid of the triangle at a height h above the plane. At a point vertically above C, at the same height h above the plane, a horizontal force F is applied to the board in the direction from B to C. Taking μ for the coefficient of friction at each stud, and assuming that $4h\mu < a\sqrt{3}$, find the value of F in order that the board may be on the point of slipping, and show that, if F is slightly increased, equilibrium will be broken by slipping at B and C, A remaining at rest.

212. A homogeneous solid in the form of a regular tetrahedron rests on a horizontal plane, being supported by three small studs at the corners of one of its faces, and subjected to a horizontal force F, applied to that vertex which is furthest from the plane, in a direction parallel to one of the horizontal edges. Assuming that the friction is great enough to prevent sliding, prove that the three studs cannot all remain in contact with the plane unless the weight of the solid exceeds $F\sqrt{6}$.

213. O is the centre of a rough ring which is inclined at an angle α to the horizontal. A uniform heavy rod, which is free to move about O (to which one end is attached), rests in limiting equilibrium on the circumference of the ring. If the rod is inclined at an angle θ to the diameter of greatest slope, prove that $\sin\theta = \mu \cot\alpha$, where μ ($< \tan\alpha$) is the coefficient of friction.

Replace the weight of the rod by forces acting at O and at the circumference: resolve the latter into forces in and perpendicular to the plane of the ring.

214. A uniform thin rod rests with one end on a rough horizontal plane and the other end on a rough vertical wall. The angles of friction at the ends of the rod are λ, λ' respectively. The inclination of the rod to the vertical is θ and its azimuth ϕ is the angle between the wall and the vertical plane through the rod. Prove that:

(1) ϕ cannot be $< 90^\circ - \lambda'$;

(2) if $\theta \not> \tan^{-1}(2\tan\lambda)$, the rod can rest in any azimuth which is $\not< 90^\circ - \lambda'$;

(3) if $\lambda + \lambda' > 90^\circ$, the rod can rest at any inclination, provided its azimuth is $\not< \sin^{-1}(\cos\lambda' \operatorname{cosec}\lambda)$.

Determine in the remaining cases the extreme inclination of the rod when it rests in a given azimuth.

215. A rod of length l can turn freely about one end, O, which is fixed, and the other end, P, which is higher than O, rests against a rough vertical wall distant a from O. Prove that in the position of limiting equilibrium the rod is inclined to the vertical at an angle

$$\sin^{-1}\left\{\frac{a}{l}\cdot\sqrt{\frac{l^2 - a^2\cos^2\epsilon}{l^2\cos^2\epsilon - a^2\cos 2\epsilon}}\right\},$$

where ϵ is the angle of friction.

Let A be the foot of the perpendicular drawn from O to the wall, AN the base of the wall, and PN the vertical through P. P describes a circle about A so that the friction acts at right angles to AP; the reactions at O and P are in equilibrium with W, the weight. $\therefore$ the reaction at P lies in the vertical plane APN, and its component perpendicular to this plane is zero.

Otherwise—Resolve parallel to AN and take moments about the verticals through O and A.

216. Two uniform circular cylinders of weights W, $4W$ and radii a, $4a$ are placed on a plane inclined at an angle $\cot^{-1}2$ to the horizontal, so as to touch along a common generating line, which is horizontal, the larger cylinder being uppermost. Prove that, if the equilibrium is limiting and the coefficient of friction is the same throughout, slipping is at the point of occurring at two of the lines of contact, and find the coefficient of friction.

217. AB is a heavy uniform bar, length a; the end B is a ring which slides on a fixed vertical rod; the end A is attached by a

weightless inextensible string, length l, to a fixed point C in the vertical rod; λ is the angle of friction between the ring (whose dimensions may be neglected) and the rod, and $l > a > \frac{1}{2}l$.

Show that two positions of limiting equilibrium can be found by the following construction:—A length KL is measured, equal to l, and is bisected at M. On LM a segment is made containing the angle $90° - \lambda$, and the circle completed. With centre K, radius a, a circle is drawn intersecting the former at P_1 and P_2. Lengths CB_1 and CB_2, equal to LP_1 and LP_2, are measured down the vertical rod from C. Then B_1 and B_2 are the limiting positions of B.

Carry out the construction to scale when $\lambda = 15°$, $l = 20$ cm., $a = 15$ cm., and make a sketch-diagram showing the bar in its limiting positions and the forces then acting.

State the length B_1B_2 as accurately as your drawing allows.

218. Four rough equal uniform spheres are placed in contact, three of them being on a rough horizontal plane and the fourth on top of the other three, each sphere touching the rest. Show that, if equilibrium is on the point of being broken simultaneously between the upper sphere and the lower spheres and also between the lower spheres and the plane, then one coefficient of friction is four times the other, and find their values.

219. A circular ring is supported on the surface of a rough sphere by three short studs placed at intervals of 120° on the circumference of the ring. The line joining the two lower studs is horizontal and the highest point of the sphere lies outside the ring. Show that, if all the studs are about to slip when the plane of the ring makes an angle α with the vertical,

$$2 \tan\beta \{3 \sin(\alpha - \theta) \cos(\theta + \beta) + \sin\alpha \cos\beta\}$$
$$= \sqrt{4 - 3\sin^2\theta}\, \{6 \cos\alpha \cos\theta \cos(\theta + \beta) - 2 \sin\alpha \sin\beta\},$$

where β is the angle of friction, and 2θ is the angle subtended by the diameter of the ring at the centre of the sphere.

220. A homogeneous solid, in the form of a regular tetrahedron, rests on a rough horizontal plane, being supported at the three corners on the plane, and is under the action of a force at the remaining corner, which is directed parallel to one of its horizontal edges. Prove that the resultant of the frictions at the supports passes through the centre of the circle on which they lie, and find the vertical reactions of the supports.

221. A uniform cardioidal disk lies on a rough plane, which is inclined to the horizon at an angle α. The disk can turn freely about a pin through its pole, and its axis makes an angle β with the line of greatest slope when equilibrium is on the point of being broken. Prove that the coefficient of friction is $\frac{3}{4}\tan\alpha \sin\beta$ and that the pressure on the pin acts along the axis of the disk.

222. A uniform rod of length $2a\sin\alpha$ has its ends on a semicircle of radius a whose axis of symmetry is vertical and whose convexity is

downwards. Prove that, if μ is the coefficient of friction, and if there is a position of limiting equilibrium with the rod inclined at an angle θ to the horizon, then $\mu \cos\theta = (\cos^2\alpha - \mu^2 \sin^2\alpha) \sin\theta$.

223. A body whose weight is 10 units is placed on a plane inclined at 25° to the horizon; the coefficient of friction between the body and the plane is 0·6. Find graphically in magnitude and direction the forces which must be applied in the plane to drag the body along the plane in directions making angles of 33° with the upward and downward-drawn lines of greatest slope respectively. State the magnitudes of the forces and their inclinations to the line of greatest slope.

224. Three uniform heavy rods DA, DB, DC of the same material are connected by a smooth hinge at D and rest with their lower ends A, B, C in contact with a rough horizontal plane, and a weight W hangs freely from D. The rods are inclined to the vertical at angles $\alpha_1, \alpha_2, \alpha_3$, and the dihedral angles between the vertical planes through them are A_1, A_2, A_3. Find which rod will be the first to slip as W is gradually increased.

If equilibrium is broken by the rods BD and CD slipping at the same instant, find the coefficient of friction.

225. Assuming that, when a rough plane joint is in limiting equilibrium, the pressure between pin and socket increases continually according to some law (the same on each side) from a point of minimum pressure to the point diametrically opposite, where the pressure is a maximum; show that the line of resultant action between the bars thus jointed together cannot in any case intersect the pin of the joint unless the ratio of maximum pressure to minimum pressure exceeds $(2a+\pi)/(2a-\pi)$, where $a = \cot\theta + \theta$ and θ is the angle of friction.

226. A square window sash weighing 30 lb. slides vertically in grooves. From the two upper corners sash cords are carried over pulleys and carry two counterpoises each of 15 lb. Show in a diagram the forces acting on the sash when one of the sash cords breaks, and find the least coefficient of friction between sash and grooves that will keep the sash from sliding down, if all other friction may be neglected.

227. (i) A rod placed on a uniformly rough table is pulled gently by a string attached to one end in a direction perpendicular to the rod and in the plane of the table. Find the point about which the rod begins to turn.

(ii) If a circular ring is pulled in the direction of a tangent, prove that it will begin to turn about the other end of the diameter through the point of contact, and find, in terms of the coefficient of friction, the ratio between the force and the weight of the ring when motion is on the point of taking place.

For (i) assume the point sought to be distant k from the C. G. Take moments about the C. G. and resolve perpendicular to the rod. Show that $k = a(\sqrt{2}-1)$, if $2a$ is the length of the rod. For (ii) see § 158, Ex. 14.

228. A circular lamina of radius a, resting on a rough horizontal table, is pulled by a force in its own plane so that the lamina is on the point of motion. If the point about which it begins to turn is at a distance ka $(k<1)$ from the centre, prove that the line of action of the force is at a distance ma from the centre, where

$$3mk\int_0^K \mathrm{cn}^2 u\,\mathrm{dn}^2 u\,du = \int_0^K \mathrm{dn}^4 u\,du \text{ (mod. } k).$$

See § 158, Ex. 14.

229. A body, resting on a rough horizontal plane, is acted on by forces parallel to that plane which are sufficiently great to overcome the friction. Prove that the axis about which it begins to move is such that the moment about it of all the forces, frictions included, is a minimum; and the condition that the forces are just sufficient to move it is found by equating to zero the least value thus found.

A uniform elliptic lamina rests on a rough horizontal plane; find the magnitude and point of application of the least force which will cause it to begin to rotate about a focus.

See § 158, Ex. 14.

230. A heavy lamina is placed in a given position resting on three given fixed surfaces α, β, and γ. The surface γ is smooth, but α and β are rough, and their coefficients of friction with the lamina keep changing owing to the melting of some paint with which they are coated. If the equilibrium, however, remains limiting at α and β, prove that the different points about which the lamina tends to rotate lie on a certain conic.

231. A hollow cylinder of radius a and weight W is placed on a rough horizontal plane in contact with a rough vertical wall, and on it a hollow cylinder of radius b $(<a)$ and of the same density is placed, also in contact with the wall. If equilibrium is possible, determine in terms of a, b, and W the frictions and normal reactions at the points of contact of the lower cylinder with the horizontal plane and of the upper cylinder with the wall and with the lower cylinder, the reaction between the lower cylinder and the wall being zero; and prove that the coefficient of friction of the upper cylinder with the wall must be not less than unity.

232. A uniform rod rests with one end against a rough vertical wall, and the other end on a rough horizontal floor, the angle between the rod and the wall being 20°, and the angle which the projection of the rod on the wall makes with the vertical being 10°. Supposing that equilibrium is about to be broken by the rod slipping on the wall, find the coefficient of friction between the rod and the wall.

233. A rectangular piece of flexible cloth is on the point of slipping over the smooth edge of a rough horizontal table; prove that the overhanging side, which is parallel to the edge of the table, is at a depth $a\sin\alpha/(\sin\alpha+\cos\alpha)$, where a is the breadth of the cloth and $\tan\alpha$ the coefficient of friction.

234. A hollow cone, the equation of whose surface is $x^2/a^2+y^2/b^2 = z^2/c^2$, is fixed with its axis vertical and vertex downwards. A uniform heavy rod, of any length, is placed with its ends on the inner surface, which is rough, so as to be parallel to the axis of y. Prove that it will rest wherever put, provided that the coefficient of friction is $\not< c/a$.

If PQ is the rod, friction (F) acts along tangents to the cone at P, Q lying in vertical planes. The equations of such a line at $P(x, y, z)$ are $(\xi-x)/z\tan^2\alpha = (\eta-c)/0 = (\zeta-z)x$; and the pressure, R, at P acts along the normal $(\xi-x)/x = (\eta-c)/c = (\zeta-z)/(-z\tan^2\alpha)$. If the direction cosines of these lines are p, q, r; l, m, n, those of the corresponding lines at Q are $p, -q, r$; $l, -m, n$. Resolve these forces and the weight along Ox and Oz and obtain the relation

$$\frac{F^2}{R^2}\tan^2\alpha = \left(1-\frac{c^2}{z^2\tan^2\alpha}\right)\left(1-\frac{c^2\cos^2\alpha}{z^2\tan^2\alpha}\right),$$

an equation for z which always has real roots. If $\mu < \cot\alpha$, the extreme value for F/R is μ. Substitute in the relation found and we get the greatest possible value of z. If $\mu > \cot\alpha$, z may have any value, for F/R never reaches its extreme value.

235. A uniform solid right circular cone whose vertical angle is less than $90°$ rests at a point A of its rim on a rough horizontal plane with the generator through A vertical. Find the magnitude, direction, and point of application of the least force which, acting at a point of the generator, will keep the cone in equilibrium. Find also the least value of the coefficient of friction in order that the cone may rest in the proposed manner.

236. Show that the work done in slowly extracting a cork, of length l and radius r, from the cylindrical neck of a bottle is $\pi\mu Pl^2 r$, where μ is the coefficient of friction, assuming the pressure per unit of area between the bottle and the unextracted part of the cork to be constant and equal to P.

A heavy plug in the shape of a frustum of a cone exactly fits a conical hole of the same size, the common axis being vertical. The vertical angle of the cone is 2α and the radii of the circular bases of the frustum are a and b. The normal reaction per unit of area being supposed constant, show that the moment of the least couple that will twist the plug is

$$\tfrac{2}{3}\mu W(a^2+ab+b^2)/(a+b)\sin\alpha,$$

where W is the weight of the plug and μ is the coefficient of friction.

The work done $= 2\pi r P\mu \int_0^l (l-x)\,dx$.

The moment $= \mu R\int_a^b y\,.\,2\pi y\,dy\cot\alpha$, where $W = R\sin\alpha \times$ area of surface of plug.

237. Two buckets are connected by a rope which passes one and a half times round a rough horizontal axle placed above the mouth of

a well. If the depth of the well is h feet, the weight of the rope per foot w, and the weight of the buckets when full and empty W and W' respectively, calculate the coefficient of friction that there may just be no slipping when the empty bucket is at the top and the full bucket at the bottom of the well and the axle held fixed. Find the couple which must be exerted in any other position of the buckets to prevent the axle turning, and hence or otherwise calculate the work done in raising the full bucket and lowering the other.

Show also that, if the axle gets jammed and cannot turn, so that the rope must be made to slide round the axle, the work done is increased in the ratio

$$(W+W'+wh)(2W-2W'+wh):2W'(W-W').$$

238. An infinite number of infinitesimal spheres are placed in a row on the concave side of an arc AB of a rough equiangular spiral in a vertical plane; the curvature of the arc diminishes from A to B; the tangent at A is horizontal and makes an angle 2α with the tangent at B. Show that the spheres will be just on the point of slipping down if the coefficient of friction between a sphere and the spiral is $\tan\alpha$, and that between any two spheres is infinite, and if the angle of the spiral is $\frac{1}{2}\pi-\alpha$.

239. A rigid triangular framework ABC, formed of three uniform bars, is suspended from a fixed point O by strings attached to the middle points D, E, F of the bars. Show that the tensions of the strings OD, OE, OF are to one another as $OD\,.\,EF : OE\,.\,FD : OF\,.\,DE$.

240. Three bars AB, BC, CD, jointed at B and C, are placed in a vertical plane, BC being horizontal and above the line AD, which is also horizontal, the ends A and D being fixed by smooth horizontal pins; the joints A and C are further connected by a bar, and the weights of all the bars are negligible. A vertical load of 112 kg. weight is applied at the middle point of AB, a vertical load of 504 kg. at the middle point of BC, and 126 kg. at the middle of CD; given the lengths $AB=15$, $BC=7$, $CD=13$, $DA=21$, calculate the stress in the bar AC.

241. Three uniform rods AB, BC, CD, each weighing 1 lb. and of length 5 in., are freely jointed together at B and C, and rest in a vertical plane upon a smooth horizontal table at A and D. Two fine light strings AC, BD, each of length 8 in., keep the framework in equilibrium when a mass of 6 lb. is placed on BC at a distance $1\frac{1}{5}$ in. from C. Find the tensions of the strings.

242. Three rods, of lengths 3, 4, 5 in., form a triangle. A weight W is hung at the middle point of the 4 in. side, and the whole is hung up by a string attached to the 3 in. side at a point 2 in. from the right angle.

Find the reactions at the joints, neglecting the weights of the rods, and treating each joint as a point.

243. Two unequal uniform heavy rods CA, CB are hinged at C, and rest in a vertical plane with their extremities A, B on a smooth

horizontal table, their middle points being connected by a weightless wire EF.* Find, in terms of the weights W, W' of the rods and the angles of the triangle ABC, the reactions at A, B, C and the tension of the wire EF.

Result. Reaction at $A = \frac{1}{2} W + \frac{1}{2}(W + W') \sin A \cos B \operatorname{cosec} C$; tension in $EF = (W + W') \cos A \cos B \operatorname{cosec} C$.

244. A derrick crane consists of a fixed vertical post and a beam or jib of the same length jointed to the post just above its foot. A chain of half the length of the post or jib connects their upper ends. Another chain fastened to the post one-tenth of the way up passes over the end of the jib and supports a load of 2,000 lb.

The weights of the post and jib are each 1,000 lb., and act one-third of the way up; friction and the weight of the chains may be neglected. Find, by a graphical or an analytical method, the magnitude and line of action of the resultant of the forces which act across the section of the post at the ground level.

245. Three jointed light rods form a triangle, one joint of which is connected with a point on the opposite rod by a string whose tension is given. Find the forces along the sides of the triangle in terms of the lengths of these sides, and of the string and the distances of the point of connexion of the string from the adjacent joints.

246. A bay of a cantilever bridge is represented in skeleton by two equal isosceles triangles ABC, DEF, formed of freely jointed bars, with their bases AB, DE vertical and supported at the lower ends A, D, and having their vertices C, F connected by a rigid horizontal platform which is under tension T; the joints A, B, D, E are stayed by similar bays on the other side. Find expressions for the stresses in the various members due to this tension T and a load W placed at a given point P on CF.

247. Of three uniform rods AB, BC, CA the two last are equal, both in length and in weight. They are freely jointed so as to form a triangle ABC. This is hung up by a thread passing under C. Show that the stress at A on the rod CA acts along a line which when produced backwards meets the rod BC.

248. A uniform rod of length $2c$ is supported by two light bars, each of length $2a$, jointed to it at common extremities, and having their other ends hinged to two points distant $2c$ apart at the same level, the whole system being in a vertical plane, and the light bars being supposed to *cross* each other freely. Prove that, if $a^2 < 2c^2$, the rod can rest in a position inclined to the horizontal.

249. Three uniform rods AB, BC, CD, each of weight w, are connected by smooth joints at B and C, and are supported by strings at A and D. Draw a force diagram, and determine the magnitudes and directions of the reactions at the joints in terms of w and the inclinations of the strings to the vertical. Also show that the reactions can only be equal if the strings are equally inclined to the vertical.

250. A tripod consists simply of three equal legs, VA, VB, VC, jointed freely at V, the length of each leg being l. It stands symmetrically on a smooth horizontal plane ABC. At three points D, E, F, one-quarter of the way down the legs, inextensible weightless strings, each of length $l/8$, are attached, and their other ends fastened together to support a weight equal to the aggregate weight of the tripod. Find, by the principle of virtual work or otherwise, the angles the legs make with the vertical in equilibrium, correct to 1°.

If in a similar tripod the points D, E, F were three-quarters of the way down the legs, and the strings three-quarters of the length of the legs, and able to pass freely through slots in the plane, the weights being as before, show that the tripod would then rest at all angles.

251. A framework consists of three light rods, jointed at their ends, in the form of a triangle ABC in which $AB = 10$ in., $BC = 8$ in., and $CA = 6$ in. The framework is suspended with AB horizontal and above C by means of two vertical strings attached to A and B, and a load of 50 lb. is suspended from C. Find, by means of a stress-diagram, the tensions in the strings and the stress in the bar AB. Is this bar in a state of thrust or tension?

Result. 32 and 18 lb. wt.; 24 lb. wt. thrust.

252. In a triangle of rods ABC, connected by smooth joints at A, B, C, each rod is connected with the opposite joint by a string perpendicular to the rod, and the tension of each string is proportional to the length of the rod to which it is attached. Construct a diagram of the tensions of the rods and the stresses of the joints, and show that, if AD is the string attached to BC and O the orthocentre of ABC, the tension of BC is proportional to $2\,AO-OD$.

253. AB, ED are light strings, each of length a, attached to fixed points A, E in the same horizontal. BC, CD are uniform equal rods, each of length $2\,a$, freely jointed at C and supported at B and D by the strings. If the rods are perpendicular when in the symmetrical position of equilibrium, find the length AE.

254. Three rods AB, BC, CD, of which AB, CD are of equal length a, and all three are of the same uniform thickness, are freely jointed at B, C, and rest in equilibrium with AB, CD symmetrically supported by two smooth pegs P, Q in a horizontal line, at a distance c apart. Find the height of BC above PQ in the position of equilibrium, and if $BC = b$, prove that $b < c < \dfrac{2a^2+2ab+b^2}{2a+b}$.

Result. $\frac{1}{2}(c-b)\tan\theta$, where $2a^2\cos^3\theta = (2a+b)(c-b)$.

255. Forces proportional to the sides BC, CA, AB of a triangle ABC act in those sides in the senses indicated by the order of the letters. Construct the funicular of which one side is the line joining the middle points of AC, AB, the stress in that side being *tension* of

amount equal to the force in the side BC; and find the stresses in the remaining sides.

256. A quadrilateral is formed by four light bars freely jointed at their extremities. It is in equilibrium under the tensions of two strings which join the two pairs of opposite vertices. Prove that the tensions of the strings are as the harmonic means of the segments into which they are divided by one another.

257. Two equal uniform heavy rods AD, CB, each of weight W, are freely jointed at their ends to two other equal uniform rods AB, CD, each of weight W', which cross one another; and the joints B, D are connected by a rod. A load, w, is suspended from A, and the framework is in equilibrium in a vertical plane, with the points B, D on a smooth horizontal plane. Show that the stress in the rod BD is equal to $(W + W' + w) \cos ADB \cos ABD \operatorname{cosec} DAB$.

258. AB, BC, CD, DA are four uniform rods, freely jointed at A, B, C, D to pins which are joined by strings AC, BD in tension. The whole is lying on a smooth horizontal table. If the thrusts in AD and BC are inversely proportional to the lengths AD and BC, show that either A, B, C, D lie on a circle or AD, BC are parallel.

259. $ABCD$ is a plane quadrilateral. Given forces P, Q act in AB, BC respectively. Determine graphically the forces R, S which must act in CE, DA in order that the four may reduce to a couple. (All forces act in the senses indicated by the letters.)

Let AB be P units of length. Produce AB to B' so that $BB' = AB$. In BC (produced if necessary) take C' so that the length $BC' = Q$ units. Complete the parallelogram $BB'HC'$. Draw HK parallel to CD and BK parallel to AD. Hence HK ($= R$ units), KB ($= Q$ units) satisfy the given condition.

260. A parallelogram $ABCD$ of uniform rods of equal weight freely jointed, in which a shorter side AD is fixed in a horizontal position, is supported by a light string, which joins DB and is vertical. Show that the tension in the string is equal to twice the weight of a rod, and find the force of compression in the rod BC.

261. Four equal uniform heavy rods loosely jointed together form a square. Two opposite corners are connected by a string, and the square hangs from one of these corners. Find the tension of the string in terms of the weight of a rod, and the reactions at the corners.

262. A rhombus is formed of four equal uniform rods smoothly jointed together, each rod being of length a and weight W, and the system rests symmetrically with its two upper sides in contact with two smooth pegs at the same level at a distance apart equal to $2c$; a weight w is hung from the lowest point; prove that the inclination α of the sides of the rhombus to the horizontal is given by the equation
$$a(4W + 2w)\cos^3\alpha = c(4W + w).$$

263. A rhombus $PQRS$ is formed by joining freely four equal uniform rods, each of weight W. A fifth uniform rod LM of half the length and weight of one of the former rods is freely jointed at its ends to the middle points of QR and RS. If the system is suspended from the hinge P, find in terms of W the stress at each of the hinges at L, M. *Result.* $\frac{1}{2}W\sqrt{30\frac{1}{3}}$.

264. $ABCD$ is a rhombus formed by equal uniform rods hinged freely at their ends A, B, C, D. The middle points E, F of AB and BC are connected by a string of less length than a rod. The whole is suspended from A. Show that the tension of the string is equal to the weight of the four rods; and find the stress at the hinge C.

265. AB, BC, CD, DA is a framework of light rods freely jointed at A, B, C, D. Points E, F in AB, AD respectively are connected by a string in a state of tension T; points G, H in BC, CD respectively are connected by a light rod; show that the rod is in a state of thrust T', where

$$\frac{T}{EF}\frac{AE \cdot AF}{AB \cdot AD} = \frac{T'}{GH}\frac{CG \cdot CH}{CB \cdot CD}.$$

Show how to draw a force diagram from which the stresses at the joints can be determined.

266. $ABCD$ is a plane quadrilateral. Determine graphically the ratios of the forces which, acting along its sides, would be in equilibrium.

P, Q, R are three points (not collinear) in AB, BC, CD. Find a point S in DA, such that, if PQ, QR, RS, SP are light rods freely jointed at P, Q, R, S, they will be kept in equilibrium by forces acting at the joints along the sides of $ABCD$.

267. Four weightless bars freely jointed at their extremities form a plane quadrilateral, and pairs of points on opposite sides are joined by two strings in a state of tension. By applying the theory of geometrical homography or otherwise, find the directions of the reactions at the joints; discuss the case also where only one string is tight.

268. $ABCD$ is a framework of four uniform rods freely jointed at A, B, C, D and such that AB is greater than but parallel to CD; and BC, CD and AD are equal. The system is suspended freely from a fixed point O by two equal strings fastened to A and B. Find the tensions of these strings, the stresses along the rods AB, CD, and the reactions at the hinges C and D, in terms of inclinations of OA, AD to the horizontal and of the weights of the rods.

269. A quadrilateral of freely jointed rods is in equilibrium under forces proportional to the lengths of the rods acting normally outwards at their middle points. Prove that a circle can be circumscribed to the quadrilateral, and that the stresses at the joints act along tangents to the circle.

270. A pentagon $ABCDE$ is formed of rods freely jointed together at the vertices, and the four rods AB, BC, CD, DE are equal and uniform. The rod AE rests on a horizontal plane and the system is in equilibrium with C vertically above F, the middle point of AE. Prove that, if G is the foot of the perpendicular from B on AC,

$$AF \cdot CF \cdot AB^2 = AC^3 \cdot BG.$$

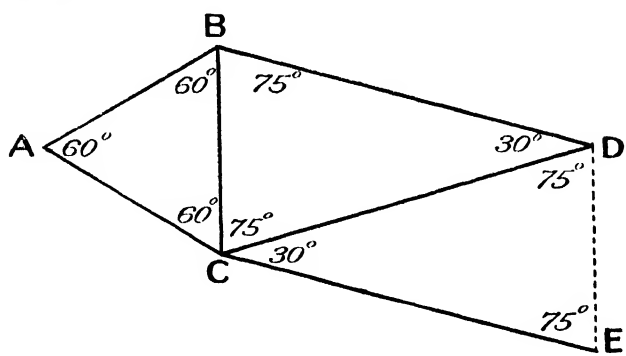
271. Five equal uniform rods, each of weight W, are freely jointed together to form a pentagon $ABCDE$, which is suspended from the joint A and maintained in the shape of a regular pentagon by two strings joining A to C and D. Show that the tension of either string is $2W \cos 18°$.

272. A framework of five jointed weightless bars AD, DC, CB, AC, BD is set up in a vertical plane by fixing the joints A and B at two points at the same level, so that the figure $ABCD$ is a trapezium with CD parallel to and above AB; weights W_1 and W_2 are supported at the joints C and D. Taking AD and CB to be of equal lengths, and CD to be shorter than AB and to be under a given tension T, determine graphically the stresses in the remaining bars.

273. A frame, in the form of a regular hexagon $ABCDEF$, is stiffened by diagonal members AD, FD, CE. The frame is supported in a vertical plane by means of vertical pressures at the ends of the lowest member AB, which is horizontal; and there are loads w_1 at F and w_2 at D $(2w_2 > w_1)$. Draw the force-diagram, and determine the forces in all the members of the frame where $w_1 = 3$ tons, $w_2 = 2$ tons.

274. Six light rods AB, BC, CA, BD, CD, CE are freely jointed at their extremities so as to form the framework shown in the figure, the points D, E being freely hinged to two fixed points in a vertical line. Determine the stress in each rod and the reactions at D and E when a weight W is attached to the joint A, assuming that the whole system is in a single vertical plane.

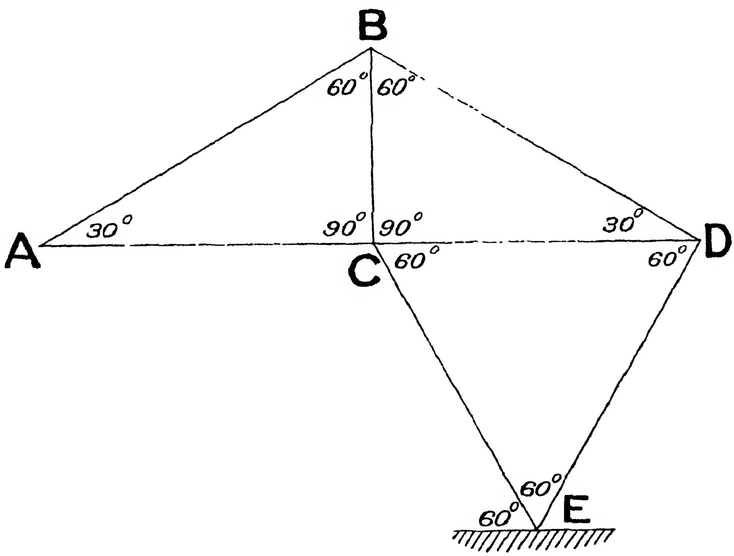
Result. Stresses in $AB = W$, in $AC = W$, in $BC = W(\sqrt{3}-1)$, in $BD = \frac{1}{2}W\sqrt{6}(\sqrt{3}-1)$, in $CD = \frac{1}{2}W(\sqrt{6}+\sqrt{2})$, in $CE = 2W\sqrt{2}$: reaction at $D = W\sqrt{11-2\sqrt{3}}$, at $E = 2W\sqrt{2}$.



275. A regular hexagon $ABCDEF$ composed of six equal and uniform rods each of weight W, freely jointed at their extremities, is just balanced in a vertical plane. The longest rod AB is fixed horizontally, and two rigid weightless wires connect A with the middle point of CD and B with the middle point of EF. Prove that the thrust of each wire is $\frac{1}{2} W \sqrt{39}$.

276. Seven light bars are freely jointed so as to form the framework shown in the figure, and rest in a vertical plane with A hinged to a fixed point and E resting against a smooth horizontal plane, a weight W being attached to C. Find the stresses in all the bars, marking with a double line those which are in a state of thrust.

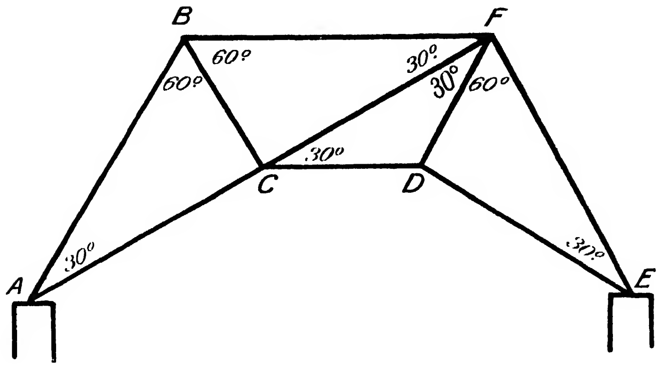
Result. Stresses in BA, BC, BD each $= \frac{2}{3} W$; in EC, ED each $= 2 W/3\sqrt{3}$; in $AC = W/\sqrt{3}$; in $CD = 4 W/3\sqrt{3}$.



277. $ABCDEA$ is a framework of seven rods of equal weight $2 W$, such that $ABCE$ and $EBCD$ are both parallelograms; the framework is suspended by vertical strings attached to A and D, AD being horizontal; the vertical through B divides AE in the ratio of $1:5$; draw a diagram completely representing the tensions and compressions of the rods, and show that if a weight $6 n W$ is attached to the point which divides BC in the ratio $1:5$, the tensions and compressions of the rods AE, ED, BC are each increased in magnitude by $W n \cot AEB$ and those of BE, EC are unaltered.

Replace the weight of each rod by a weight W at each end, and construct the force diagram. Also replace $6nW$ by $5nW$ at B and nW at C. In the force diagram produce the lines representing the

stresses in CD and AE to points which are in the same vertical and a distance a apart.



278. The framework of nine light rods freely jointed together represented by the above figure rests on smooth horizontal supports at A and E, which are at the same level, and is loaded with weights of 40 cwt. at C and 24 cwt. at D. Show that the pressures on the supports are 34 cwt. at A, and 30 cwt. at E, and determine the stress in each rod.

279. Four rods AB, BC, CD, DA jointed together form a square. AE, DE, BF, CF are four more rods, the first two being jointed at E and the last two at F. All these eight rods are equal. The joints E and F lie outside the square and E is fixed. BD is a ninth rod inserted in the framework. A weight W is suspended from F. The rods are light and all the joints are frictionless. Draw a force diagram exhibiting the stresses in the rods, and find the stresses in ED, BD.

280. Equilateral triangles BDE, CDF are described externally on two adjacent sides of a square $ABDC$, and nine weightless rods, joined together at their extremities by smooth hinges, are placed so as to coincide with the eight lines of the figure and the diagonal BC. Three equal weights, W, are attached to the joints A, E, F, and the whole is suspended by two vertical strings attached to B and C, BC being horizontal. Draw a force diagram showing the stresses in the several rods, and prove that that in BC is $\frac{1}{2}W\sqrt{3}$.

281. AB, AB', BB', AD, DC, CB, AD', $D'C'$, $C'B'$ are nine equal light rods freely jointed together; AC, AC' are two equal rods jointed at A, C and A, C', of such lengths that the angles ABC, $AB'C'$ are each 150°, the parallelograms being outside the triangle ABB'. The system is suspended from C and C', CC' being horizontal, and equal weights W are suspended from B, B', D, D' and a weight $2W$ from A. Determine, graphically or otherwise, the tensions in the rods.

282. A framework of coplanar jointed light rods AB, BC, CA, AD, DB, AE, EC is such that ABC is equilateral and AD, AE perpen-

dicular to AB, AC respectively, and the angles ABD, ACE are each $60°$; three equal forces P in equilibrium are applied in the plane of the framework at the joints A, D, E, whose directions meet in O; find the forces acting along the rods in terms of P.

283. A_1A_2, A_2A_3, ..., A_nA_1 are light rods freely jointed at A_1, A_2, A_3, ..., A_n and acted on by forces along A_1O, A_2O, ..., the rods and O being in one plane. When the system is in equilibrium, show that the stresses in the rods are proportional to the distances from O of the polar reciprocals of A_1A_2, A_2A_3, ... with respect to any circle with centre O.

284. If the sides of a force polygon are drawn in a given order, find the funicular, derived from it, for which the sum of the squares of the stresses in its sides is least.

285. A light tetrahedral framework $ABCD$ is built of six rods loosely jointed in threes at A, B, C, D. A string joining P in AB to Q in CD is in a state of tension T and induces stresses in the rods. Determine the reactions at A and B on the rod AB, and show that the rod CA is in a state of stress whose magnitude is

$$T.AC.BP.DQ/PQ.AB.CD.$$

(Prove that the action at A on the rod AB must lie in AQ and that at B in BQ. Their resultant must be T in QP. Show that the action at A on AB is $T.QA.BP/AB.PQ$. This is the resultant of stresses in AD and AC.)

286. A polygon is formed by projecting any polyhedron on a plane, and another is formed by projecting on the same plane the reciprocal of the first polyhedron with regard to any paraboloid of revolution whose axis is perpendicular to the plane. Show that if the sides of either polygon are replaced by thin bars smoothly jointed at the ends, this frame will be in equilibrium if the stress in each bar is proportional to the corresponding side of the other polygon.

287. A uniform plate $ABCD$ has the form of an equilateral triangle ABD and an isosceles triangle CBD, whose angle BCD is $120°$, on opposite sides of the same base BD. A heavy particle, whose mass is two-thirds that of the plate, is attached to the plate at C. Show that the C.G. of the system lies on BD.

288. The sides (taken in order) of a uniform quadrilateral lamina are 6, 4, 3, 5 inches respectively, and the first side is parallel to the third. Find the distance of the C.G. from the first side.

289. $ABCD$ is a lamina in the form of a trapezium, AB, CD being the parallel sides, $2a$, $2b$ their respective lengths. Prove that, if P and Q are the middle points of AB and CD, and R is the middle point of PQ, the C.G. of the lamina coincides with that of masses at P, Q, R respectively proportional to a, b, $2(a+b)$.

290. Show that the C.G. of a uniform solid tetrahedron coincides with that of a geometrically equal tetrahedral wire frame, occupying

the same position, whose edges are formed of six wires of equal weights; also that it coincides with the C.G. of a geometrically equal thin tetrahedral shell, occupying the same position, whose faces are of equal weights and each of uniform thickness.

291. Prove that the C.G. of a ring-shaped solid, bounded by the surfaces of two spheres, and by two planes each perpendicular to their common axis, is on this axis, at a distance from the point half-way between the planes which is equal to the square of the distance between the planes divided by twelve times the distance of this point from the radical plane (plane of intersection) of the two spheres.

292. A solid lead plummet is in the form of a right circular cone, 3 inches high, surmounting a hemisphere of 1 inch radius. Prove that the plummet, if suspended from a point in the rim of the common base of the cone and the hemisphere, will hang with the generator of the cone opposite to this point horizontal, provided there is suspended from the vertex of the cone a mass whose weight is $\frac{1}{80}$ of that of the plummet.

293. A bowl of uniform thin material in the form of a hemisphere (of radius a) is closed by a plane circular lid, of the same material and thickness, which has a circular hole (of radius c) cut from it, the centre of the lid being on the edge of the hole. Show that if it is placed on a horizontal table the plane of the lid will make an angle $\tan^{-1}\frac{c^3}{a^3}$ with the horizontal plane.

294. A uniform wire $ABCD$ is bent at right angles at B and C so that it forms three sides of a rectangle. Show that, if $BC = AB\sqrt{2}$, the distance of the C.G. of the wire from A or D will be equal to AB.

295. A uniform straight bar AB, 5 ft. long, is bent at a point C, distant 3 ft. from A, so as to form an angle ACB. Prove that, when the bar hangs freely from the end A, a plumb-line suspended from A will cross CB at a distance of $5\frac{1}{7}$ in. from its middle point.

296. A uniform prism of square section $ABCD$ rests on a horizontal plane with $ABCD$ vertical and AB lowest, and a wedge-shaped piece is cut off by a plane parallel to the horizontal edges passing through the middle point of BC and cutting AB in P. Prove that when the wedge is removed the prism will fall over unless AP is greater than $AB\,(\sqrt{6}-2)$.

297. A uniform lamina is made up of a rectangle $ABCD$ and an isosceles triangle ABE, whose equal sides are AB, AE. Prove that it can stand on the edge BC on a smooth table, provided the area of the triangle bears to that of the rectangle a ratio not greater than $\sqrt{3}:2$.

298. A uniform triangular lamina ABC which has the angle A obtuse can turn freely about A, which is fixed. When weights w_1, w_2, w_3 are suspended successively from B the system rests in equilibrium with the circumcentre, the incentre, and the orthocentre

in the vertical line through A. Find the weight of the lamina in terms of w_1, w_2, w_3.

299. From the vertices A, B, C of a triangle, perpendiculars AD, BE, CF are drawn to the opposite sides; particles m_1, m_2, m_3 are placed at D, E, F respectively, and their C. G. coincides with the orthocentre; prove that $m_1 : m_2 : m_3 :: \sin 2A : \sin 2B : \sin 2C$.

300. (i) G is the C. G. of a triangle ABC formed of rods of uniform thickness; the density of each of the rods BC, CA, AB varies as the distance from B, C, A respectively and is the same at unit distance; show that the areas of BGC, CGA, AGB are as

$$2b^2+c^2 : 2c^2+a^2 : 2a^2+b^2.$$

(ii) A regular tetrahedron of height h has a tetrahedron of height xh cut off by a plane parallel to the base: and when the frustum that is left is placed with one of its slanting faces on a horizontal table, it just topples over. Show that $x^3+x^2+x = 2$.

301. The C. G. of the area of a loop of the curve $r = a\cos 2\theta$ is at a distance $128\sqrt{2}\,a/105\pi$ from the origin.

302. A uniform lamina has the form of the area bounded by the curve $ay^2 = x^3$ and the straight line $x = y$. If it is freely suspended from the point corresponding to the origin, find the angle at which the straight part of its boundary is inclined to the vertical.

303. Find the C. G. of that portion of the arc of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}} = a^{\frac{2}{3}}$ which lies in the positive quadrant between the limits $x = a$ and $27x = 8a$.

304. The C. G. of the area included between two parabolas which have a common vertex and their axes at right angles lies on the common chord.

305. A wire in the form of a plane curve is such that the C. G. of any arc lies on the straight line joining a fixed point to the intersection of the tangents at the ends of the arc. Find the density at any point of the wire.

306. Find the C. G. of a circular lamina whose density varies as the distance from a straight line in its plane which does not cut the circle.

307. The abscissa at any point of a certain curve, passing through the origin, is κ times the angle which the curve makes with the axis of x, and the density at any point is proportional to the radius of curvature. If (x, y) are the co-ordinates of the C. G. of the arc joining the origin to the point (ξ, η), prove that

$$\xi x+\eta y = \xi^2+\eta^2-\kappa\eta.$$

Prove that $x = \xi-\eta\cot\psi$, $y = \eta-\kappa+\xi\cot\psi$.

308. The distance from a given point on the axis of the C. G. of a plane area symmetrical about an axis, and bounded by two

ordinates perpendicular to the axis, is a given function u of the distances x, x_0 of these ordinates from the given point; show that the equation of the bounding curve is

$$y(u-x)\exp.\left\{\int(u-x)^{-1}du\right\}=c\frac{du}{dx}.$$

309. A uniform right cone on a circular base is divided into two parts by a plane parallel to a generator. Find the C. G. of each portion.

310. A right circular cone is cut by a plane so that the section is an ellipse (centre C); prove that the C. G. of the portion of the surface cut off between the vertex and the plane lies on a line through C parallel to the axis of the cone.

311. A homogeneous solid is bounded by the surface

$$(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}+(z/c)^{\frac{2}{3}}=1;$$

prove that the C. G. of the portion of it lying in the positive octant is at the point (at, bt, ct), where $128\,t=21$.

312. Prove that the solid formed by the revolution of the cardioid $r=a\,(1+\cos\theta)$ about its axis can rest in unstable equilibrium on a horizontal plane with the axis inclined to the vertical at an angle $\sin^{-1}(5\sqrt{3}/9)$.

313. Find the C. G. of a solid hemisphere of radius a in which the density varies as the distance from the centre of the sphere.

314. A solid is bounded by half the surface formed by the revolution of the cardioid $r=a(1+\cos\theta)$ about its axis and by a plane base through the axis; prove that the distance of its C. G. from the base is $\frac{63}{128}a$, and its distance from the pole, measured parallel to the axis, is $\frac{4}{5}a$.

315. The density at each point of the solid bounded by the paraboloid $b^{-2}y^2+c^{-2}z^2=a^{-1}x$, and the plane $x=a$, varies as the square of the distance from the point $(a, 0, 0)$. Find the position of the C. G. of the solid.

316. The density at any point of an ellipsoid varies as x^3y^2z, where x, y, z are the co-ordinates referred to the principal planes. If an octant is hung up by the centre, find the inclinations of the principal axes to the vertical line.

317. The density at any point x, y, z of the solid ellipsoids whose surfaces are determined by the tangential equations,

$$b\,m^2+c\,n^2+2f\,mn+2g\,nl+2h\,lm+2ul+2v\,m+2w\,n+1=0,$$
$$b'm^2+c'n^2+2f'mn+2g'nl+2h'lm+2u'l+2v'm+2w'n+1=0,$$

is $\phi(x)$. Show that the masses are in the same ratio as the volumes.

318. The ends of a uniform heavy chain are attached to two fixed points A, B. When the chain is hanging in equilibrium the tangents to it at its extremities make angles $\tan^{-1}(3/4)$, $\tan^{-1}(5/12)$ with the

horizontal in opposite senses. Prove that the inclination of the straight line AB to the horizontal is $\cot^{-1}(6 \log 3)$.

(Show that the horizontal distances of A and B from the vertex of the catenary are $c \log 2$ and $c \log \frac{3}{2}$.)

319. The weight per foot run of a telegraph wire is w lb., the length of wire between the posts is l ft., a weight W lb. is hung at the middle, and the wire sags x ft. in the middle. Prove that the tension at the posts is

$$w\left\{\frac{l^2}{8x}\left(1+\frac{2W}{lw}\right)+\frac{1}{2}x\right\} \text{ lb.}$$

320. A suspension bridge has a span of 450 ft., and is uniform, weighing 5 tons per foot of length. It is supported by two wire cables, the lowest points of the cables being 40 ft. below their ends at the towers; the weight of the cables may be neglected. By considering the equilibrium of either half of the bridge under (1) its weight, (2) the horizontal pull at the middle of the cables, (3) the pull at the end of the cables, find the pull at the middle of a cable and at the end, and the inclination of the cable at the end.

321. A heavy string 70 in. long hangs over two smooth pegs not in the same horizontal line. The parts which hang vertically are respectively 20 and 25 in. long. Prove that the vertex of the catenary formed by the part between the pegs is 28 in. along the string from one end and 42 from the other.

322. A uniform string is suspended from two points in the same horizontal and a mass (whose weight is equal to that of a length $2l$ of the string) is fastened to the middle of the string. If the portions of the string make an angle 2α with one another, show that the vertex of either of the catenaries is at a horizontal distance $l \tan\alpha \sinh^{-1}(\cot\alpha)$ from the vertex of the catenary in which the string would hang if the mass were removed.

323. A suspension bridge is formed of a cable, AOB, whose ends A and B are fixed at the tops of two vertical posts, AB being a horizontal line of length 500 ft.; O is the lowest point of the cable and is 50 ft. below AB. The cable is loaded uniformly throughout its length at the rate of 2 tons per horizontal foot length. To the tops A, B of the pillars are also attached two back-stays each inclined at 45° to the vertical. What must be the tension in each back-stay if the resultant force acting on each pillar at its top is vertical, and what is the tension of the cable at each extremity?

324. A string of length $2l$ hangs in a vertical plane with its ends attached to points A and B on the same level, the distance AB being equal to $2a$. A smooth heavy ring R can slide on the string and rests in a given position, equilibrium being maintained by a horizontal force P applied at R. C is the middle point of AB and the vertical through R meets AB in N. Show that the ratio of P to W is equal to $(l^2-a^2) . CN/l^2 . RN$.

Also if equilibrium is maintained by a force in the line CR, find an expression for its value.

325. A wire rope is fastened to two points at different levels and is horizontal at the lower point. The differences of level of four points P_1, P_2, P_3, P_4 on the rope at equal horizontal distances a apart is observed, and h_{12} is the difference of level of P_1 and P_2, h_{23} and h_{34} having similar meanings. Prove that the tension at the lowest point of the rope is $wa/\cosh^{-1}\{(h_{12}+h_{34})/2h_{23}\}$, where w is the weight of the rope per unit length; and if the length of the rope is known, find the height of the upper end above the lower, and the tension at the upper end.

326. A uniform heavy chain AB has the end A fixed, and the other end B can be held at any point on a horizontal line through A. When AB is 5 ft., the depth of the lowest point of the chain below the ends is one-quarter of its length. Through what distance must B be moved in order that the depth of the lowest point of the chain below the ends may be two-fifths of its length? *Result.* 2 ft.

327. Two chains, of linear densities ρ_1 and ρ_2 and of lengths l_1 and l_2, are tied together at one pair of ends (C), and suspended by the other pair of ends from two points A, B at the same level. If the junction is at the lowest point of the curve assumed by the compound chain, prove that C is at a depth $\{(l_1^2\rho_1-l_2^2\rho_2)/(\rho_1-\rho_2)\}^{\frac{1}{2}}$ below AB.

328. A uniform chain of length $3s$ has its ends attached to small smooth rings which slide one on each of two fixed rods in the same vertical plane. The rods slope downwards from their point of intersection, making angles of 45° and 60° with the vertical respectively. To a smooth light ring sliding on the chain a weight equal to that of a length $(\sqrt{3}-1)s$ of the chain is attached. Determine the ratio of the lengths of the parts of the chain on the two sides of the ring when the system is hanging in equilibrium.

329. A tight rope, one yard of which weighs one pound, is stretched between two points distant 30 ft. apart at the same height from the ground. The sag of the rope is one foot. Find, to the nearest pound, its tension at the points of support.

330. A uniform chain 10 ft. long, weighing 2 lb. per foot, is attached at one end to the top of a vertical post fixed in the bed of a stream, its other end being attached to the stem of a boat. The stream exerts on the boat a horizontal force equal to the weight of 15 lb., and the vertical height of the top of the post above the stem of the boat is 6 ft. Prove that the horizontal distance between the stem and the post is $15 \log_e (5/3)$ ft.

(Prove that $c = 15$, and that the arcual distance of the top of the post from the vertex of the catenary $= 91/4$.)

331. AB is a piece of uniform string suspended at A and B, the horizontal through the lower of these points B cutting the string

again in B'. A portion of similar string, whose length is equal to twice the depth of B or B' below A, is suspended from two points in the same horizontal line, whose distance apart is twice the horizontal distance between A and B', and it is found that the tension at either point of support is equal to the weight of the former string. Prove that, if s is the length of the given string, d the depth of B below A, and h the horizontal distance between A and B,

$$h = \sqrt{(s^2-d^2)}\log(\sqrt{2}+1).$$

332. (i) A finite length of weightless thread is loaded with an infinite number of equal heavy particles at equal distances along it, the total weight per unit length being w, and has its ends fixed on the same level. Exhibit a force diagram for determining the tension at any point; from inspection of it obtain the form of the intrinsic equation of the catenary; and deduce the Cartesian equation, when the length is $2l$ and the distance apart of the fixed ends is $2a$.

(ii) Obtain also the intrinsic equation when the particles instead of being all equal are of weights varying as their distances along the thread from its middle point; and prove that the tangent of the thread's inclination to the horizon at a fixed end is

$$\tan^2\left(\frac{1}{2}am\frac{2a}{c}\right), \text{mod.} \frac{1}{\sqrt{2}},$$

where c is given in terms of l and a by

$$l = c\tan\left(\frac{1}{2}am\frac{2a}{c}\right).$$

For (ii) show that $\tan\phi \propto \int_0^s s ds$. Put $s = c\sqrt{\tan\phi} = c\tan\psi$.

Then $2x = c\int_0^{2\psi} d(2\psi)/\sqrt{1-\frac{1}{2}\sin^2 2\psi}$.

333. $ABCDE$ is a heavy uniform string of length $4l$. C, B, D bisect AE, AC, CE respectively. Particles of weight equal to a quarter of that of the string are attached at B and D, and one equal to half the weight of the string is attached at C. The ends A, E of the string are fastened to two points $2b$ apart in the same horizontal. Show that the different parts of the string are portions of equal catenaries, the common parameter c satisfying the equation resulting from the elimination of t and t' between

$$t+t' = b, \quad 4l = 3l\cosh(t/c) + \sqrt{9l^2+c^2}\sinh(t/c),$$
$$2l = l\cosh(t'/c) + \sqrt{l^2+c^2}\sinh(t'/c).$$

334. A heavy uniform string of weight $2w$ and length $2l$ hangs freely from two fixed points B, C in the same horizontal plane distant $2a$ from each other. A second similar string of length l and weight w has its lower end fastened to the middle point D of BC, and the upper end A can move freely along a horizontal line bisecting BC at right angles. If A is moved slowly along

this line from the position of equilibrium, prove that the tangent to the second string at D will be horizontal, when and only when ABC are the vertices of an equilateral triangle; and find an expression for the work done in bringing A to this position.

Prove that in the position of equilibrium the tangents at D are horizontal. If there is a position other than the first in which the tangent at D to the string DA is horizontal, then the tangents at D to DB and DC must also be horizontal. Prove that the parameters of the three catenaries are equal and (from the equilibrium of D) that the tangents at D to DA and DB include an angle $\frac{1}{3}\pi$. The work done = work done in raising the weights of the catenaries through the vertical distances traversed by their C.G.'s

$$= w\left[c_1\left\{\cosh(a/c_1) - a/l\right\} + \tfrac{1}{2}l - \tfrac{3}{2}c_2\left\{\cosh(2a/c_2\sqrt{3}) - 2a/l\sqrt{3}\right\}\right],$$

where $$l = c_1 \sinh(a/c_1) = c_2 \sinh(2a/c_2\sqrt{3}).$$

335. One end of a light inextensible string of length $2a$ is attached to a point on a smooth straight fixed wire, which is inclined at an angle of 45° to the horizontal. The other end of the string is attached to a ring of mass m, which slides on the wire. A particle of mass $2m$ is attached to the middle point of the string. Show that the inclination of either portion of the string to the wire when the system is in equilibrium is $\cot^{-1} 2$.

336. Two pegs are at a distance 20 in. apart in the same horizontal line. One end of a string is tied to one peg and the other end carried over the other peg, the part between the pegs carrying a weight of 1 lb. which can slide freely on it. Find the tension of the string when the weight is 1, 2, 3, 4, 6, 8, 10 in. respectively below the line joining the pegs, and represent the results by means of a graph.

Show that this graph has two asymptotes, and indicate their position.

337. An inelastic string is suspended from two fixed points at the same level and hangs under gravity; prove that, whatever the distribution of density along the string may be, the tangents of the inclinations to the horizontal of the tangents to the curve of the string at the two points of support are inversely proportional to the horizontal distances of the C. G. from the supports.

338. A rope is stretched between two posts 50 ft. apart on the same level. In wet weather it is observed to sag in the middle 3 in. below the points where it is tied to the posts, and in dry weather it sags as much as 2 ft.

Assuming the shape of the curve formed by the rope to be parabolic, find the pull of the rope on the posts in the two cases, taking the rope to weigh 5 lb.

339. An endless cord passes over two small rough pegs, which are at the same level and at a distance $2a$ apart. If the tangents at the pegs to the two catenaries in which the cord hangs are inclined

to the horizontal at angles of 30° and 60°, prove that the distance between the lowest points of the catenaries is

$$\frac{a}{\log(2+\sqrt{3})} - \frac{(2-\sqrt{3})\,a}{\sqrt{3}\log\sqrt{3}}.$$

Determine also, neglecting the weight of the cord in contact with the pegs, the coefficient of friction between the cord and the pegs, if the equilibrium is limiting in the case considered.

340. A heavy string fastened to a point on a curve in a vertical plane lies along its concave side, the free extremity of the string being at that point of the curve at which the tangent is horizontal; find the form of the curve when the pressure at any point varies as the curvature.

341. A long heavy uniform inextensible string is hung over the rough convex arc of a cardioid $r = a(1-\cos\theta)$, fixed in a vertical plane, the tangent at the cusp being horizontal, and rests in limiting equilibrium with a portion at each end hanging vertically. If the lengths of these free portions are such that the one nearer the cusp is $e^{-\pi\tan\alpha}$ times the other, prove that the angle of friction α is given by the equation $36\sqrt{3}\,(2e^{-\pi\tan\alpha}+1) = 16\cot\alpha - 35\sin 2\alpha$.

342. A string of mass M and length $2l$ under the action of gravity hangs in the form of part of a cycloid, the tangents at the two ends being equally inclined to the horizon and at right angles to each other. Show that the mass per unit of length at any point of the string is $Ml^2\rho^{-3}$, where ρ is the radius of curvature of the string at that point.

343. A heavy uniform chain has its ends fastened at two points on a smooth cone of semivertical angle 45°, whose axis is vertical and vertex upwards, and rests in equilibrium on the convex side of the cone. If R is the distance of any point P of the chain from the vertex of the cone, and ϕ the angle made by the chain with the generating line through P, find the equation in terms of R and ϕ to the curve assumed by the chain, and deduce its radius of curvature at P.

See § 299. Take the vertex as origin, Oz vertically downwards. Let r, ψ be the co-ordinates of the projection on the plane zx of a point $P(x, y, z)$. Then $T = w(h-z)$, $Tr^2 d\psi/ds = \text{constant} = wk^2$, where $w =$ the weight per unit length of the string, and $z = r$. Hence $d\psi/dz\sqrt{2} = k^2/z\sqrt{\{z^2(h-z)^2-k^4\}}$. Now suppose the surface of the cone developed into a plane and R, θ the polar co-ordinates of P. Then $\psi = \theta\sqrt{2}$, $R = z\sqrt{2}$, and

$$\tan\phi = R\,d\theta/dR = b^2/\sqrt{\{R^2(R-2a)^2-b^4\}},$$

where $b = k\sqrt{2}$, $h = a\sqrt{2}$. We can deduce for the developed catenary the relation $(2a-R)p = b^2$.

In order to obtain the radius of curvature we may employ the formula $1/\rho^2 = \Sigma\,(D^2x)^2$, where $D \equiv d/ds$.

From the equation of the cone, $x^2+y^2=z^2$, we find $\Sigma(xDx)=2zDz$ and $\Sigma(xD^2x)=2zD^2z+2(Dz)^2-1$. If then P is the pressure per unit length due to the cone, the equations of equilibrium of the string are $D(TDx)+Pl=0$, $D(TDy)+Pm=0$, $D(TDz)+Pn+w=0$, where $l\sqrt{2}=\cos\psi$, $m\sqrt{2}=\sin\psi$, $n\sqrt{2}=1$. Multiply these equations by x, y, z and add: we get

$$T\{2zD^2z+2(Dz)^2-1\}+Pz/\sqrt{2}+wz=0.$$

Now from the equations of equilibrium we get

$$Dz=dz/d\psi\div ds/d\psi=\sqrt{\{(h-z)^2z^2-k^4\}}/(h-z)z\sqrt{2}.$$

Differentiate zDz with respect to s and thus obtain the value of P.

We have further $(h-x)D^2x=DxDz-(P\cos\psi)/\sqrt{2}$, &c. Square and add, noting that $Dx\,.\cos\psi+Dy\,.\sin\psi+Dz=2Dz$.

344. A string rests on the upper side (which is entirely concave) of a smooth curve in a vertical plane. The mass m per unit length of the string varies from point to point, the law of variation and the form of the curve being such that the pressure varies directly and the tension inversely as m. Prove that the length and the mass of the portion of the string between the lowest point and the point where the tangent makes an angle ϕ with the horizon are respectively constant multiples of

$$(1-\cos\alpha)^2\left(\frac{1}{\sin\alpha}\frac{d}{d\alpha}\right)^2\left\{\frac{1}{\sin\alpha}\log\frac{\sin\frac{1}{2}(\alpha+\phi)}{\sin\frac{1}{2}(\alpha-\phi)}\right\}$$

and

$$(1-\cos\alpha)\left(\frac{1}{\sin\alpha}\frac{d}{d\alpha}\right)\left\{\frac{1}{\sin\alpha}\log\frac{\sin\frac{1}{2}(\alpha+\phi)}{\sin\frac{1}{2}(\alpha-\phi)}\right\},$$

where α is the inclination to the horizon of the curve's asymptote.

Put $R=mgc$, $mT=\mu^2g$. The equations of equilibrium reduce to $\mu^2dm/ds=-m^3\sin\phi$, $\mu^2d\phi/ds=m^2(\cos\phi-c)$. Since

$$d\phi/ds>0,\ \therefore\ c<1=\cos\alpha\text{ (say)}.$$

Obtain $m=m_0(\cos\phi-\cos\alpha)/(1-\cos\alpha)$, and hence express $ds/d\phi$ as a function of ϕ; which integrate.

345. A heavy string whose weight is W rests on the concave side of the cycloid $s=4a\sin\phi$ whose axis is vertical and vertex downwards, one end being fixed at the point $\phi=\sin^{-1}\sqrt{2/3}$, and the other end free at the vertex; show that the tension at the fixed point is $W/\sqrt{6}$, that the string is always in contact with the curve, and that the resultant pressure on the curve is $W/\sqrt{2}$.

(The horizontal and vertical components of the resultant pressure are respectively $8\,mga\sqrt{2}/3\sqrt{3}$ and $4\,mga/3\sqrt{3}$.)

346. The tangent at the highest point A of an arc AB of a rough equiangular spiral in a vertical plane is horizontal, and the tangent at B is inclined at an angle 2α to the horizontal; show that if the angle of the spiral is $\frac{1}{2}\pi-\alpha$, and the angle of friction is α, a heavy uniform chain of length equal to the arc will rest upon it in equilibrium.

347. Prove that a string can rest in equilibrium in the form of an ellipse under forces (per unit length of the string) $\mu r_1^{\frac{1}{2}} r_2^{-\frac{1}{2}}$ and $\mu r_1^{-\frac{1}{2}} r_2^{\frac{1}{2}}$ from the foci respectively, r_1, r_2 being the focal distances of any point of the string, and that the tension is $2\mu(r_1 r_2)^{\frac{1}{2}}$.

Use the equation $r_1 + r_2 = 2a$ and the relations

$$p_1/r_1 = p_2/r_2 = b/\sqrt{r_1 r_2}.$$

348. The force at any point in the plane of xy is along and proportional to the ordinate. Prove that the form of rest of a uniform flexible and inextensible string with two points fixed in the plane is such that the rectangle under the orthogonal projections of the radius of curvature on the ordinate and of the ordinate on the radius of curvature is constant; and that, if a^2 is this constant taken positively and $(0, c)$ the co-ordinates of the vertex, the equation of the curve is $c = y \operatorname{cn}(a^{-1}x)$, mod. $\frac{1}{2}a^{-1}(4a^2 - c^2)^{\frac{1}{2}}$.

We have $dT/ds = \mu y \sin\phi$, $T/\rho = \mu y \cos\phi$. Hence $T = T_0 \sec\phi$ and $\rho\cos\phi \times y\cos\phi = \text{const.} = a^2$ say. If $y_1 = dy/dx$, $y_2 = dy_1/dx$, then $a^2 y_1 y_2/\sqrt{1+y_1^2} = yy_1$: whence $2a^2(\sqrt{1+y_1^2} - 1) = y^2 - c^2$, and

$$x = a\int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}, \text{ where } y = c\sec\phi \text{ and } 4a^2k^2 = 4a^2 - c^2.$$

349. If the fixed ends of a catenary of uniform strength are originally in the same horizontal line, and one of them is moved horizontally, prove that the chain may rest in the form of the curve $\tanh s/a = \operatorname{sn} x/a$.

Will the ends of the chain have been brought nearer together, or farther apart?

350. Find the form of the closed curve in which a uniform flexible inelastic string will rest under a central force varying at any point directly as the perpendicular from the centre to the tangent and inversely as the square of the central radius vector, it being given that λ is the constant moment of the tension about the centre, μ the force at unit distance, and a the distance of an apse from the centre of force. *Result.* A conic with latus rectum $2h/\mu$, where $h = Tp$.

351. A uniform endless chain rests upon a smooth right cone, whose axis is vertical and vertex upwards. Show that, at either of the points where the chain is horizontal, its tension is equal to the weight of a piece of the chain whose length is the vertical distance of the other point below the vertex of the cone.

See Ex. 343. Here $\tan\phi = \infty$. $\therefore$ $R^2 - 2aR \pm b^2 = 0$, and $R_1 + R_2 = 2a$ or $z_1 + z_2 = h$, if z_1, z_2 are the depths below the vertex of the cone of the points at which the chain is horizontal. Thus $T_1 = w(h - z_1) = wz_2$ and $T_2 = w(h - z_2) = wz_1$.

352. A heavy string is hung over the rim of a smooth fixed hemisphere of radius unity with its axis vertical and vertex downwards, so that one end of the string is at the vertex. If the string is

in equilibrium, find what portion of it is not in contact with the hemisphere and prove that the total length of the string is

$$\tfrac{1}{3}\pi + \tfrac{1}{4}(4 + \sqrt{15} - \sqrt{3}).$$

Find the point at which the pressure of the hemisphere vanishes, and consider the catenary which begins at that point.

353. A uniform flexible string rests with every point of its length in contact with the concave side of a smooth catenary, the axis of which is vertical and the vertex downwards. One end of the string lies at the vertex, and the other end is fixed at a point P. The length of the string is such that its weight is equal to $\sqrt{3} \times$ the tension at P. Prove that the tension at P is equal to the resultant pressure on the curve.

354. A string of infinite length passes through two small smooth rings, and is acted upon by a force, which always tends from a given fixed point and varies inversely as the n^{th} power of the distance from that point. Show that, if $n > 1$, the part of the string between the two rings is a portion of a curve whose equation is $r^{n-2} = a^{n-2} \sin(n-2)\theta$. (Prove that $p \propto r^{2n-1}$.)

355. A uniform inelastic string with fixed ends is repelled from a given plane by a force everywhere proportional to the mass acted upon and the distance from the plane. Prove that it assumes the form of a plane curve, and that the tension along it varies as the product of the distances from two planes parallel to the given plane.

356. A heavy string fastened to a point on a smooth curve fixed in a vertical plane lies along its concave side with its free end at the point of the curve at which the tangent is horizontal; find the form of the curve such that the pressure at any point varies as the curvature.

357. A heavy string is in equilibrium in a vertical plane under the action of gravity and a central force repelling with an intensity proportional to the distance. Show that the resultant of the tensions at the ends of any element of the string passes through a fixed point O; and find the pedals with respect to O of the catenary of uniform cross-section and the catenary of uniform strength.

358. A piece of uniform string with its ends fixed is at rest in the form of a parabola under a force from the focus. Find the law of force; and prove that if the ends are moved to other fixed points (the law remaining unaltered), the new figure of equilibrium will be a curve of the system $p^2 = r(cp + c')^2$.

Use the equation $p^2 = ar$, and find $\mu/2mr^{\frac{3}{2}}$ for the law of force. By integration we deduce for the general case $T = \mu/r^{\frac{1}{2}} + \text{const.} = h/p$.

359. A smooth cylinder, whose cross-section is the complete cycloid generated by a circle of radius a, is placed with its generators horizontal and the generator through the vertex highest. A smooth string is attached to two points on the highest generator and rests

on the cylinder, just reaching the generator through one of the cusps. Show that the equation of the curve it assumes may be written in the form

$$s+z=\frac{4a}{k}\left\{E(u)-k^2\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u}\right\};$$

where $2uak'^2=kz$, the distance between the points of attachment is $4ak'^2K/k$, and s is the arc and z the horizontal distance from the lowest point of the string.

360. A heavy string is placed in equilibrium on a smooth sphere; prove that if an hyperboloid of one sheet is described having as generators the tangents to the string at any two points, and the line of action of the weight of the part of the string between these two points, then it will pass through the centre of the sphere.

361. A heavy uniform inelastic string PQ is laid on a rough curve in a vertical plane; show that it will rest if the inclination of the chord PQ to the horizontal is less than $\tan^{-1}\mu$, where μ is the coefficient of friction.

(Let $\mu=\tan\alpha$. Draw a chord PS inclined at an angle α to the horizon, and let A be the point at which the tangent is parallel to PS. The tension, T, at a point R distant s from P along the string, when this is about to slip downwards, is given by

$$Te^{-\mu\psi}=w\int e^{-\mu\psi}(\sin\psi-\mu\cos\psi)\,ds,$$

where ψ, $\equiv\tan^{-1}dy/dx$, is the inclination of the tangent at R to the horizontal, (x, y) being the co-ordinates of R referred to horizontal and vertical axes. If η is the vertical distance of R below PS, $\eta=y-\mu x$, and, since $T=0$ at P and at Q,

$$\int_P^Q e^{-\mu\psi}d\eta=0.$$

If the tangent at R makes an angle $\theta\ (=\alpha-\psi)$ with PS, we deduce that

$$\int_P^Q e^{\mu\theta}d\eta=0.$$

Now from P to A $d\eta$ is negative and θ is positive, and from A to S $d\eta$ is positive and θ is negative. Also $e^{\mu\theta}$ is greater throughout PA than throughout AS. Hence, unless the upper limit θ of the integral is beyond S, it is impossible for the last integral to vanish: so not all the available friction is required to keep the string at rest unless Q is beyond S.)

362. One end of a uniform chain of length $a+b$ is attached to a fixed point at a height h about a fixed horizontal plane on which the length a of the chain rests in a straight line. The whole chain is in a vertical plane, and a is so great that the chain is in limiting equilibrium. Prove that $b^2=h^2+2\mu ha$, where μ is the coefficient of friction between the chain and the plane. (The chain reaches the plane at the vertex of the catenary.)

363. A man wants to support a mass of 10 cwt. by using a few turns of a rope round a rough horizontal cylinder under which he is standing. How many turns will be necessary in order that he may not have to exert a greater force than one of 75 lb. weight, the coefficient of friction being $\frac{1}{3}$?

364. Round three cylinders, each of radius a and weight W, passes a light elastic band, which lies in that plane perpendicular to their axes which contains their centres of gravity; the modulus of elasticity of the band being $2W/\sqrt{3}$. One of the cylinders is held with its axis horizontal and the other two are supported below it by the band so as to have their axes in the same horizontal plane and to remain in contact with the upper cylinder and with each other. Show that the natural length of the band cannot be greater than

$$2a\left(\frac{2k^2+1}{k^2+1}+\frac{1}{\mu}\log\frac{k^4-1}{4k-4}\right),$$

where μ is the coefficient of friction, and k is written for $e^{\frac{\mu\pi}{3}}$.

Let C be the highest point of the band, A one of the points where it leaves the upper cylinder to meet a lower cylinder at B and to remain in contact with it as far as the lowest point D, and let D' be the lowest point of the other lower cylinder. For the natural length to be greatest the tension, T, must be as small as possible. Between A and B the least value of $T=2w/\sqrt{3}=\lambda$; $\therefore$ natural length of $AB=a$. The natural length of the portion $AC=a\mu^{-1}\log\frac{1}{2}(1+k)$; of $BD=a\mu^{-1}\log\frac{1}{2}(1+k^2)$; of $DD'=2a/(1+k^{-2})$.

365. A uniform inelastic heavy string is fastened at one end to the highest point of a rough sphere of radius a, and rests in limiting equilibrium, with each element in contact with the sphere and about to move at right angles to itself. Prove that the lower end cannot be free; and that, if this end is on the equator, the tension there cannot be less than $\mu wa/(1-\mu)$, where μ is the coefficient of friction and w the weight of unit length of the string.

366. A rough surface is formed by the rotation of an inverted catenary, of parameter c, about its axis of symmetry which is vertical. A piece of string of weight w per unit length is tied to a weight W which rests at the vertex. The tangent at the other end of the string makes an angle α, greater than the angle of friction, with the horizontal; show that the greatest weight which can be tied to this end without disturbing equilibrium is $[\mu We^{\mu\alpha}+wc(e^{\mu\alpha}-\sec\alpha)]/(\sin\alpha-\mu\cos\alpha)$, μ being the coefficient of friction between the rough surface and both the string and the weights.

See § 196. From the equations

$$R-T/\rho=w\cos\psi,\ dT/ds=\mu R-w\sin\psi,\ s=c\tan\psi,$$

we obtain $Te^{-\mu\psi}=A-wc\,e^{-\mu\psi}\sec\psi$, where $\mu W=A-wc$ (when $\psi=0$). Let T', R' refer to the lower end when a weight W' is attached and there is limiting equilibrium. Then

$$T'+\mu R'=W'\sin\alpha,\ R'=W'\cos\alpha.$$

367. A light string rests on a rough surface in a state bordering on motion, and the form of the string is a geodesic. Prove that (i) the friction of the string at any point acts along the tangent to the geodesic, and (ii) the ratio of the tensions at any two points is equal to exp. $\mu\phi$, where ϕ is the sum of the infinitesimal angles turned through by a tangent which moves from one point to the other.

368. One end of a uniform heavy rough string, of length s, is attached to a fixed point O and the other to a point P, whose co-ordinates referred to horizontal and vertical axes through O are x, y. The string is threaded through a number of minute uniform heavy spherical beads, smooth outside but rough in the bore, and these beads, which extend the whole length of the string but do not press against the points of support, are in limiting equilibrium. If μ is the coefficent of friction between beads and string, prove that the locus of P may be obtained by eliminating c between the equations

$$2\,\mu\tan^{-1}\frac{s^2-y^2}{cs} = \log\frac{s+y\tanh(x/c)}{s-y\tanh(x/c)},\quad s^2-y^2=c^2\sinh^2(x/c).$$

[The beads are so small that the 'line of pressure' coincides with the string.]

369. AB is a heavy elastic string, and P is a point on it such that $AP:PB::\sqrt{2}-1:1$. When the string hangs freely from A as the point of support, prove that the stretchings of the parts of the string above and below P are equal.

370. Four rods AD, BD, CD, OD of negligible weight are placed so that A, B, C, D are the corners of a regular tetrahedron, A, B, C being in a horizontal plane and D above the plane, while O is the centroid of the triangle ABC. The rods are of the same material and cross-section, and are slightly compressible, and the ends A, B, C, O are fixed. A weight W is supported at D. Assuming that Hooke's law holds, prove that the thrust in the rod OD is $\frac{1}{5}(2\sqrt{6}-3)\,W$, and that in either of the other rods is $\frac{1}{15}(4\sqrt{6}-6)\,W$.

(If T=the thrust in OD, and T_1=that in each of the other rods, prove, by resolving vertically for the equilibrium of W, that $W=T+T_1\sqrt{6}$. Also, if AD and OD are shortened by lengths x_1 and x, we have by geometry $AD\,.\,x_1 = OD\,.\,x$ and by Hooke's Law

$$T=\lambda x/OD,\quad T_1=\lambda x_1/AD.\quad \therefore\; 2T=3T_1.)$$

371. A heavy elastic string, uniform when unstretched, is enclosed in a smooth straight tube inclined to the vertical. The string is fastened at its upper end and is allowed to stretch itself under its own weight. Prove that the ratio of the stretched to the unstretched length is $(1+2\,k/c)^{\frac{1}{2}}+1:2$, where c is the length of that portion of the unstretched string whose weight is the modulus of elasticity, and k is the difference of level of the two ends of the stretched string in the position of equilibrium.

Deduce from $(1+T/\lambda)\,dT+m_0 g\,dy=0$ the equation

$$(ds/ds_0)^2 = 1+2\,(k-s\sin\alpha)/c.$$

372. A circular disk of diameter D and mass m has n slightly elastic strings, each of natural length D, attached to equidistant points along its circumference; and the other ends of these strings are fastened to equidistant points on the circumference of a fixed horizontal circle of diameter $3D$. Show that, λ being the (very great) coefficient of elasticity, the depth below the circle of the position of equilibrium of the disk is $D\sqrt[3]{2mg/n\lambda}$.

373. An elastic string of length $2l$ hangs in the form of a catenary of uniform strength so that the tension at any point is wa, where w is the weight per unit length at the point in the stretched state and a is a constant. If the unstretched length $2l_0$ is such that $l_0 = a\tanh(l/2a)$, prove that the weight per unit length in the unstretched state is proportional to $(a^2+s_0^2)(a^2-s_0^2)^{-2}$, where s_0 is the length of the arc measured from the strongest point of the string.

374. A particle of weight W is fastened to the middle point of a light elastic string (modulus λ). The ends of the string are fastened to two points in the same horizontal plane whose distance apart is the length of the unstretched string. Prove that each half of the string is inclined to the horizon at an angle θ given by $\tan\theta - \sin\theta = W/2\lambda$.

If the weight of the particle is just sufficient to increase by one-half the length of the string when hanging fastened to one end, the other being fixed, find θ to the nearest degree.

375. An elastic string, whose modulus of elasticity is equal to the weight of a length l of the unstretched string, is placed on a smooth horizontal table with its ends fastened to two fixed points A, B on the table. Each element is acted on by a horizontal force kmy at right angles to AB, where m is the mass of the element and y is its distance from AB. Find the relation between the y of any element P and the angles the tangents to the string at P and A make with AB, given that the tension at A is equal to the weight of a length c of the string.

Show also that, if the curve along which the string lies is revolved about AB, the area of the surface so traced out is

$$2\pi cg\{(c+2l)\sin\alpha + c\cos^2\alpha\log(\tan\alpha+\sec\alpha)\}/kl,$$

where α is the angle between AB and the tangent to the string at A.

From $(1+T/m_0gl)\,dT + km_0y\,dy = 0$ we get

$$ky^2 = gl(1+c/l)^2 - gl(1+\cos\alpha/l\cos\phi)^2.$$

The area

$$=\int_A^B 2\pi y\,dy/\sin\phi = 4\pi gc\cos\alpha\int_0^\alpha(\sec^2\phi + c\cos\alpha/l\cos^3\phi)\,d\phi/k.$$

376. AB is a uniform light elastic string of natural length $4a$ and modulus λ, C, D points on the string such that, when it is straight and unstretched, $2AC = CD = 2DB$. The extremities A, B of the string are attached to two points in the same horizontal plane whose distance apart is $4a$, and equal weights W are attached to the string

at C and D. Show that, when the system is hanging in equilibrium, the total extension of the string is a $(\sec\theta - \cos\theta)$, where θ is the inclination of the end portions of the string to the horizontal.

Also find the ratio $W:\lambda$ in the case where $\theta = 60°$.

377. An elastic string, of natural length $2a$, is fastened to the extremity of a horizontal diameter of a smooth circle of radius a, which can turn about its centre in a vertical plane. The circle is turned very slowly until the natural length of the part of the string which hangs freely is $\frac{3}{8}a$; prove that the angle turned through is given by

$$\log \tan (3\pi/8) + \log \tan (\pi/8 + \phi/4) = 13/2\sqrt{2},$$

the modulus of elasticity of the string being one-sixteenth of its weight. See Ex. 382.

378. A flexible, very slightly extensible, string, uniform when unstretched, hangs under gravity with its ends attached to fixed points at the same level and at a distance apart equal to the natural length. Prove that the tension at the lowest point is approximately equal to $\frac{1}{2}(\frac{1}{3}EW^2)^{\frac{1}{3}}$, where E is the modulus of elasticity and W is the weight of the string.

With the notation of § 199, if $2l =$ the unstretched length of the string, the required tension $= m_0 gc$, $W = 2m_0 gl$, $E = m_0 ga$, $x = l$ when $s_0 = l$; $\therefore l = lc/a + c \log (l/c + \sqrt{1 + l^2/c^2})$ and

$$e^{l/c}(1 - l/a) = l/c + \sqrt{1 + 1^2/c^2} \quad q.p.$$

Expand and show that $6c^3 = al^2 \quad q.p.$

379. The ends of an elastic string, uniform when unstretched, are fastened to two points A and B in a smooth horizontal plane on which the string rests in equilibrium, under the influence of a force perpendicular to AB, while the coefficient of elasticity λ is equal to the component of the tension parallel to AB. The force per unit length of unstretched string at a distance y from AB is

$$\lambda \frac{y^3}{b^4}\left\{1 + \left(\frac{3}{2} - \frac{1}{2}\frac{y^4}{b^4}\right)^{-\frac{1}{2}}\right\},$$

where b is the maximum value of y. Determine the Cartesian equation of the curve in which the string lies and show that

$$b\{\Gamma(1/4)\}^2 = 2\pi^{\frac{1}{2}} AB.$$

380. The cross-section of a heavy flexible and elastic string made of homogeneous material, whose density ρ and modulus of elasticity E are unaltered by stretching, is $a \cosh ps$ at distance s from its centre when unstretched, a and p being constants and $p > g\rho/E$, where g is the acceleration of gravity. Its unstretched length being $2l$, find the distance between the points in the same horizontal plane to which its ends must be fastened for it to be uniformly stretched when it hangs freely, and show that it then hangs in the curve

$$(p - g\rho/E)\, y = \log \sec (p - g\rho/E)\, x.$$

[In estimating tensions, do not neglect to consider the variation of the cross-section.]

381. A perfectly flexible inextensible string is in equilibrium under forces per unit length which are derivable from a potential V. Prove that the form of the string is such that $\int V\,ds$ taken along the string has the minimum value consistent with the given length of the string.

A heavy elastic string hangs with its ends fixed at two points, distant $4a\tan\gamma$ apart in the same horizontal line, in the form of a parabola of latus rectum $4a$, the modulus of elasticity being n times the tension at the vertex of the parabola. If the string is cut at the vertex, and either part is allowed to take up its position of equilibrium, show that the 'stretching' of this part is

$$a\left[\tan^2\gamma - 2n(\sec\gamma - 1) + 2n^2\log\{(n+\sec\gamma)/(n+1)\}\right].$$

382. The upper end of an elastic string, whose modulus of elasticity is k times its weight, is fastened to a point on the rim of a rough wheel whose axis is horizontal, and the lower end is fastened to a weight equal to K times the weight of the string. Initially the string hangs freely under the action of its weight only, the weight at its lower end being supported by a horizontal plane, and the diameter through its upper end is horizontal. The wheel is now turned slowly so as to stretch the string. Prove that the weight will not begin to rise until the weight of the string not in contact with the wheel is reduced to $\sqrt{(K+k)^2+1+2k}-K-k$ times the weight of the string, and find the angle through which the wheel must be turned, the friction being sufficient to prevent any slipping of the stiing on the wheel.

Let a be the unstretched length of the string; mag its weight. The vertical distance, h, from the weight to the centre of the wheel $= a+a/2k$. If $h_0 =$ unstretched length of the vertical string when the tension at the end just $= Kmag$, we get $h = h_0+h_0^2/2ak+Kh_0/k$ and also $= a+a/2k$: whence h_0.

Let $l_0 =$ unstretched length of the vertical string when the tension at the end $= \mu mag$; then $h = l_0+l_0^2/2ak+\mu l_0/k$. The turn of the wheel through an angle $d\theta$ stretches this into a length $h+cd\theta$ and alters μ into $\mu+d\mu$. Hence we obtain $cd\theta = aUdv/k$, where

$$v = \mu+k,\ U = -v+\sqrt{v^2+2k+1}.$$

Express $d\theta/dU$ as a function of U and integrate. The limits of μ are zero and K: whence $kc\theta/a = (1-u^2)/4u-(2k+1)\log u$, where $u = \sqrt{(K+k)^2+1+2k}-k-K$.

383. An elastic string lies in a smooth tube whose form is an equiangular spiral, and is acted on by a force to the pole varying inversely as the cube of the distance. One end of the string is fixed at the point of the spiral at which $r = a$, and its natural length would extend to the point $r = 7a/10$. Show that in that position of equilibrium in which it is stretched least, it extends to $r = a/2$, provided that the modulus of the string is equal to the force on a natural length a of the string concentrated at a distance a.

384. The ends of a uniform rod of length 16 in. and mass 7 lb. are attached by similar elastic strings, each of natural length 9 in., to two fixed points, so that, when the system is in equilibrium, the rod and strings form three sides of a square. A variable couple, whose axis is vertical, now acts upon the rod so as to turn it *very* slowly round. Find the angle through which the rod has turned when it has risen through 2 in., and, in foot-pounds, the work done by the couple.

Result. 90°. Work done by couple in raising the rod = 7/6 ft. lb., in stretching the string = 4/3 ft. lb.: total 2·5 ft. lb.

385. A weightless elastic string of length a hangs vertically. One end of the string is attached to the rim of a perfectly rough wheel, whose axle is at a height a above the ground, and the other end is attached to a weight W which rests on the ground. The weight W is just sufficient to stretch the string to double its natural length. If T is the work done in slowly turning the wheel until the weight leaves the ground, and T' the additional work required to wind up all the string, show that $T = (1 - \log 2)\, T'$.

386. A flexible elastic band surrounds in the plane of their centres three equal spheres which are in contact with one another on a smooth horizontal plane. A sphere whose radius is one-third of the radius of either sphere, when placed on the top of the spheres touching them all three, causes the lower spheres to be on the point of separating. Prove that the tension of the band is one-third of the weight of the upper sphere.

387. A smooth paraboloid of revolution is placed with its axis vertical and vertex upwards; upon it is placed a heavy elastic string, of unstretched length $2\pi c$. Prove that the string, when in equilibrium, rests in the form of a circle of radius $4\pi ac\lambda/(4\pi a\lambda - cW)$, where W is the weight of the string, λ its modulus of elasticity, and $4a$ the latus rectum of the generating parabola.

388. One end of a heavy elastic string is fastened to the highest point of a smooth sphere and the other to a mass whose weight is equal to the modulus of elasticity of the string and also to the weight of a length of the unstretched string equal to the diameter of the sphere. The mass thus supported rests in contact with the sphere in a horizontal plane through the centre. Find the unstretched length of the string and the force exerted by the sphere on an element at the highest point.

389. The ends of a heavy elastic string, which lies on a smooth sphere, are fastened to two fixed points at the same level on a vertical great circle of the sphere, and the lowest point of the string is at the pole of the great circle. Obtain a differential equation connecting θ, ϕ, where the centre of the sphere is origin, θ is the angular distance from the vertical, and ϕ the azimuth measured from the vertical plane through the origin and the lowest point of the string; and show how the length of the string could be determined if its modulus of elasticity and mass per unit unstretched length are known.

390. A heavy uniform elastic string hangs in equilibrium over two smooth pegs in a horizontal plane. Prove that, if the tangent at any point at a distance x from the line of symmetry of the catenary makes an angle ψ with the horizon, then

$$\log \frac{1+\sin\psi}{\cos\psi} + k \tan\psi = \frac{x}{c}$$

(where c and k are constants), and that the free ends descend to a depth $c\,(1+\frac{1}{2}k)$ below the vertex of the catenary.

An elastic string is formed of such material that the density ρ is $\mu x \exp.(x/a)$, and the modulus of elasticity $\lambda\rho(a-x)/x$, at the point whose distance from one end of the string when unstretched is x, a being the natural length of the string. Prove that if the string is hung up by this end it will be uniformly stretched. See § 199.

391. A very slightly extensible string, which is uniform in its natural state, is suspended from two fixed points and is acted on by gravity alone. If λ is the coefficient of elasticity and w the weight of a unit length of the unstretched string, show that the form of the string is approximately given by the equation

$$y = u - \frac{w}{2\lambda} v^2 + \frac{w^2}{2\lambda^2} vu^2,$$

where $u = c \cosh x/c$ and $v = c \sinh x/c$, and c is a constant.

392. A smooth tube of small section is in the form of a semicircle of radius a whose diameter is vertical. In the tube is an extensible string (uniform when unstretched) attached at one end to the highest point. If the other end just reaches to the lowest point of the tube, and the modulus is equal to the weight of a length $8\,a/3$ of unstretched string, show that the natural length of the string is

$$2\,a\sqrt{2/5}\,F(\sqrt{3/5},\, \tfrac{1}{2}\pi),$$

and find the point of the tube at which the reaction between the string and the tube vanishes.

393. If three forces, not in the same plane, acting on a particle are represented by $l\,.\,OA$, $m\,.\,OB$, $n\,.\,OC$ respectively, their resultant is represented by $(l+m+n)\,OG$, where G is the point whose areal co-ordinates with respect to the triangle ABC are l, m, n.

394. A rigid body is acted on by three forces, each of magnitude P; their lines of action are $y=0$, $z=a$; $z=0$, $x=0$; $x=a$, $y=a$; and their directions are the positive directions of the axes, which are rectangular. Prove that they are equivalent to a force P along the axis of x, and a force $P\sqrt{2}$, whose line of action is $x=0$, $y-z=a$.

395. Three forces each of magnitude P and acting in the positive directions of the axes have as their lines of action

$$\left.\begin{aligned} y &= -a \\ z &= a \end{aligned}\right\}, \quad \left.\begin{aligned} z &= -a \\ x &= a \end{aligned}\right\}, \quad \left.\begin{aligned} x &= -a \\ y &= a \end{aligned}\right\};$$

prove that they are equivalent to a force $P\sqrt{3}$ at the origin and

a couple. Find also the magnitude and direction of the axis of this couple.

396. Forces P, Q, R act in the lines

$$(x+a)/(-l) = (y-b)/m = (z-c)/n,$$
$$(x-a)/l = (y+b)/(-m) = (z-c)/n,$$
$$(x-a)/l = (y-b)/m = (z+c)/(-n),$$

where l, m, n are direction cosines, and the senses of the forces are indicated by the signs of the denominators. Prove that, if the system is statically equivalent to a single force, $\Sigma\{P^{-1}(bm^{-1}-cn^{-1})\} = 0$.

Assuming this condition to be satisfied, find the line of action of the resultant, and show that it meets the line

$$(x-a)/l = (y-b)/m = (z-c)/n.$$

397. Six forces $P.BC$, $Q.CA$, $R.AB$, $P'.DA$, $Q'.DB$, $R'.DC$ act along the edges BC, CA, AB, DA, DB, DC of a tetrahedron. Show that if $P.P'+Q.Q'+R.R' = 0$, the six forces are equivalent to a single force or a couple; and that if they are equivalent to a couple, $P' = R-Q$, $Q' = P-R$, $R' = Q-P$.

398. Three forces act along the lines

$$x = y-z-a = 0, y = z-x-a = 0, z = x-y-a = 0.$$

Prove (1) that they cannot reduce to a couple; (2) that, if they reduce to a single force, the line of action lies on the surface

$$x^2+y^2+z^2-2yz-2zx-2xy = a^2.$$

399. $ABCD$ is a face of a cube; A', B', C', D' the other extremities of the diagonals of the cube which pass through A, B, C, D respectively. Forces of magnitudes 1, 2, 2, 7, 4 act along BC, CD, DB', $B'C'$, $C'D'$ respectively; show that they are equivalent to a single force.

400. Prove that a force P can in general be resolved into six components, one along each edge of an arbitrary fixed tetrahedron.

If the tetrahedron is $VABC$ and the given force P meets VAB, VAC in H, K respectively, prove that, if X is the component along BC, the ratio of $X.BC$ to $P.HK$ is equal to that of the projections of VHK and VBC on a plane perpendicular to AV.

401. When a system of forces is reduced to two, represented by two straight lines which do not meet and are not parallel, the volume of the tetrahedron of which these straight lines are opposite edges is known to be constant.

If the shortest distance between the two forces is one edge of the tetrahedron, show that the figure intercepted by the tetrahedron on a plane through the central axis of the system perpendicular to this shortest distance is a parallelogram whose area is $GR\,ab\,(a+b)^{-3}$, G being the central couple, R the resultant of translation, and a, b the shortest distances of the central axis from the two forces.

402. Two generators of one system of the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2 = 1$$

meet the plane $z = 0$ in the points A and C, and two generators

of the opposite system meet the same plane in the points B and D such that AB is parallel to CD. Forces, whose components parallel to the axis of z are equal, act at A, B, C, D along these generators. Show that they have a single resultant.

403. Show that any system of forces may be represented by six forces acting in the edges of a tetrahedron and proportional to those edges, and give a construction for such a tetrahedron having one vertex at a given point.

404. Forces $P_1, P_2, P_3, P_4, P_5, P_6$ act along six non-intersecting lines $L_1, L_2, L_3, L_4, L_5, L_6$, such that P_1, P_2, P_3, P_4 form a system in equilibrium, and P_3, P_4, P_5, P_6 are also in equilibrium. Show that L_1, L_2, L_5, L_6 are generators of the same kind of a conicoid.

405. If forces P, Q, R, P', Q', R' along the edges BC, CA, AB, AD, BD, CD (whose lengths are a, b, c, a', b', c') of the tetrahedron of reference of tetrahedral volume co-ordinates, have a single resultant in the line of intersection of the planes

$$l\alpha+m\beta+n\gamma+p\delta=0,\quad l'\alpha+m'\beta+n'\gamma+p'\delta=0,$$

prove that

$$P/a\,(lp'-l'p)=Q/b(mp'-m'p)=\ldots=P'/a'(mn'-m'n)=\ldots=\ldots$$

Deduce, or otherwise obtain, the two conditions that six forces acting along the edges may be equivalent to a force along the above line and a couple.

406. Show that three conditions are necessary and sufficient to secure that forces P, Q, R, P', Q', R' along the edges BC, CA, AB, DA, DB, BC, respectively, of a tetrahedron may be equivalent to a couple; and find a set of three conditions.

If the conditions are satisfied, prove that the plane of the couple makes with DA, DB, DC, respectively, angles whose sines are proportional to

$$\frac{P}{DA\,.\,BC},\quad \frac{Q}{DB\,.\,CA},\quad \frac{R}{DC\,.\,AB}.$$

Any three of the four conditions

$$P'/DA+Q'/DB+R'/DC=0,\quad R/AB-Q/CA-P'/DA=0,$$
$$P/BC-R/AB-Q'/DB=0,\quad Q/CA-P/BC-R'/DC=0,$$

are necessary and sufficient. If these conditions are satisfied, the plane of the couple passes through D and through the resultant of P, Q, R, the trilinear equation of which referred to ABC is

$$P\alpha+Q\beta+R\gamma=0.$$

407. A system of forces which is in equilibrium is such that the force acting at (x_1, y_1, z_1) has axial components (X_1, Y_1, Z_1).

If each force of the system is resolved into two forces, one parallel to, and the other perpendicular to, Oz, show that, in order that the two systems of forces parallel to Oz and perpendicular to Oz may be separately in equilibrium, we must have $\Sigma zX=0$ and $\Sigma zY=0$, i.e. in addition to $\Sigma X=0$, &c., and $\Sigma(yZ-zY)=0$, &c.

408. A system of n forces is such that a typical force acts at the point (x_r, y_r, z_r) and has components (X_r, Y_r, Z_r). Show that, if the points of action are rotated round the axis of z and the forces remain unaltered in magnitude and direction, the couple on the resultant wrench will have a stationary value when

$$\sum_{r=1}^{n}\sum_{s=1}^{n}[(x_r-x_s)(Z_rX_s-Z_sX_r)+(y_r-y_s)(Z_rY_s-Z_sY_r)]=0.$$

409. Forces whose magnitudes are constant and whose directions are fixed in space act at fixed points of a rigid body. X, Y, Z are the components, referred to rectangular axes *fixed in space*, of the force acting at that point of the body whose co-ordinates, referred to rectangular axes *fixed in the body*, are x, y, z. Prove that whatever point in the body be chosen as origin, it is possible so to choose the axes fixed in the body and the directions of the axes fixed in space that

$$\Sigma Xy=\Sigma Xz=\Sigma Yz=\Sigma Yx=\Sigma Zx=\Sigma Zy=0.$$

410. Show that a given wrench can be represented in an infinite number of ways by pairs of equal forces with shortest distance along any chosen line which cuts the axis at right angles, and that the lines of action of all such forces lie on a fixed hyperbolic paraboloid to which the axis of the wrench is normal.

411. Show that the central axes of two wrenches of pitches p and p' will intersect if

$$XL'+YM'+ZN'+X'L+Y'M+Z'N=(p+p')(XX'+YY'+ZZ').$$

If three wrenches of given pitches having three given concurrent axes are such that the pitch of the resultant wrench is given, show that the axis of the resultant wrench is parallel to some generator of a fixed quadric cone.

412. Wrenches of constant but different pitches have their axes along three mutually perpendicular intersecting straight lines. If they are equivalent to a single force, find the locus of the line of action of this force.

A wrench of *given* pitch is equivalent to a set of forces whose lines of action are generators of the same kind of a hyperboloid of one sheet. Prove that the axes of all such wrenches lie on a hyperboloid whose cyclic planes coincide with those of the given hyperboloid.

413. Four forces act along four generators of the same kind of a conicoid. Show that they cannot be equivalent to a couple unless the conicoid is a paraboloid.

(Replace the couple by an infinitesimal force, P, at an infinite distance in the plane of the couple. Then $-P$ at infinity and the four forces are in equilibrium. Show that the four generators are parallel to the same plane.)

414. Find the line of action and the magnitude of the resultant of three forces which are represented in magnitude and direction by the lines from the origin of rectangular co-ordinates to three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

415. The positive directions of the axes of co-ordinates being inclined at angles of 60°, the equations of three lines are

$$\begin{matrix} y = a \\ z = -a \end{matrix}; \quad \begin{matrix} z = a \\ x = -a \end{matrix}; \quad \begin{matrix} x = a \\ y = -a \end{matrix}.$$

A particle is acted on by three attracting forces directly proportional to its distances from these lines and along the perpendiculars drawn from it to the lines, the absolute intensity of the forces being the same. Find the position of equilibrium.

416. OA, OB, OC are three mutually perpendicular lines of lengths a, b, c respectively; complete the parallelepiped of which they are edges, and let O' be the corner diagonally opposite to O. Find the pitch of the screw to which forces represented in magnitudes, lines of action, and sense by the lines AC, CO', $O'B$ reduce.

417. $2n$ equal forces P act at the vertices of a regular polygon in the plane of xy along generators of the same system of the hyperboloid $(x^2 + y^2)/a^2 - z^2/c^2 = 1$. Show how to replace them by four equal forces, two parallel to each of the planes of xz and yz, along generators of a rectangular hyperboloid of revolution.

Let the direction cosines of the generator through $(x', y', 0)$ be (λ, μ, ν), where $\lambda/a \sin\theta = \mu/(-a\cos\theta) = \nu/c = 1/\sqrt{a^2+c^2}$. The central axis being the z-axis, the resultant force is $2nP\nu$, and the resultant couple is $\Sigma P(\mu x' - \lambda y') = 2nPa^2/\sqrt{a^2+c^2}$ in the negative direction. These must be identified with $4Q/\sqrt{2}$ and $4Qa'/\sqrt{2}$, if each of the equal forces $= Q$. Hence Q and a'. The hyperboloid is $x^2 + y^2 - z^2 = a^4/c^2$.

418. Forces acting along generators of a conicoid reduce to a single force. Show that forces of the same magnitude acting in the same sense along the corresponding generators of any confocal conicoid reduce to a single force.

Note. If (x, y, z) is any point on a generator of the conicoid whose semi-axes are a, b, c, then the point $(a'x/a, b'y/b, c'z/c)$ will lie on the corresponding generator of the conicoid whose semi-axes are a', b', c'.

419. Generators of the same system are drawn on a hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \ (a > b)$$

from points on the principal elliptic section corresponding to equal indefinitely small increments $d\phi$ of the eccentric angle, and equal forces $P\,d\phi$ act in each generator all in the direction making z increase. Prove that the system of forces can be represented by two equal forces along the generators at the extremities of the major axis of this section, each equal to

$$2\{(b^2 + c^2)/(a^2 + c^2)\}^{\frac{1}{2}} KP,$$

where K is the complete elliptic integral of the first kind, modulus $\sqrt{\{(a^2 - b^2)/(a^2 + c^2)\}}$.

The axial components of the force acting in the generator

$$(x-a\cos\phi)/a\sin\phi = (y-b\sin\phi)/(-b\cos\phi) = z/c$$

are $X = Pd\phi a\sin\phi/l$, $Y = -Pd\phi b\cos\phi/l$, $Z = Pd\phi c/l$, where $l^2 = a^2\sin^2\phi + b^2\cos^2\phi + c^2$. Show that $\Sigma X = \Sigma Y = 0$,

$$\Sigma Z = 4Pc\int_0^{\frac{1}{2}\pi} d\phi/\sqrt{a^2+c^2-(a^2-b^2)\cos^2\phi} = 4PcK/\sqrt{a^2+c^2},$$

$$\Sigma L = \Sigma M = 0,\ \Sigma N = -Pab\int_0^{2\pi} d\phi/l = -4PabK/\sqrt{a^2+c^2}.$$

Now in the second arrangement let each force $= Q$; then $X' = Y' = 0$, $Z' = 2Qc/l'$, where $l'^2 = b^2+c^2$. Equate Z' to ΣZ and N' to ΣN.

420. Any three forces X, Y, Z act along the lines whose equations are

$$\begin{pmatrix} y = -a \\ z = \ \ a \end{pmatrix},\quad \begin{pmatrix} z = -a \\ x = \ \ a \end{pmatrix},\quad \begin{pmatrix} x = -a \\ y = \ \ a \end{pmatrix}.$$

Write down the equations of their central axis.

If R is the resultant force and kR the moment about the central axis, show (1) that k must lie between $-a$ and $2a$; (2) that, for a given value of k, the central axis is a generator of a fixed hyperboloid of revolution.

The equations of the central axis are

$$\{a(Y+Z)-yZ+zY\}/X = \ldots = \ldots = k.$$

Eliminating X, Y, Z we get

$$-(k+a)\Sigma x^2 + a(\Sigma x)^2 + (k+a)^2(2a-k) = 0.$$

Take new axes so that the new x-axis is normal to the plane $\Sigma x = 0$. The surface generated is $y^2+z^2-x^2(2a-k)/(a+k) = (a+k)(2a-k)$, the generators of which must be real.

421. A body under the action of a given system of forces is kept in equilibrium by two more forces, whose magnitudes are in a given ratio, and one of which has a fixed point of application. Find the locus of the line of action of each.

422. Show that a system of forces acting along the generators of one class of a hyperboloid of one sheet cannot be kept in equilibrium by a system of forces acting along generators of the other class.

423. Prove that, if a system of forces is reduced to two forces, the shortest distance between the lines of action of the two forces meets the central axis and cuts it at right angles.

424. At every point of a fixed right line which is perpendicular to the central axis of a system of forces and meets it, a straight line is drawn representing one of two forces to which the system can be reduced; show that the locus of its extremity is $Rxy + Kz = KR$, K and R being the central couple and resultant of translation, and the central axis and fixed line being axes of z and x respectively.

425. The shortest distance between the axes of two screws is c, the angle between them is α, and the pitches are p, q. If wrenches act on these screws, show that:

(1) the axis of the resultant wrench intersects the shortest distance;

(2) the pitch of the resultant wrench is

$\operatorname{cosec} \alpha \{p \cos \theta \sin (\alpha-\theta) + q \cos (\alpha - \theta) \sin \theta \pm c \sin \theta \sin (\alpha - \theta)\}$, where θ and $\alpha-\theta$ are the angles made by the axis of the resultant wrench with those of the component wrenches.

Let the forces be R_1 along Ox, R_2 along $y = x \tan \alpha$, $z = c$. Find X, Y, Z, L, M, N. The axis of the resultant wrench is given by

$$(pR_1 + qR_2 \cos \alpha - cR_2 \sin \alpha + zR_2 \sin \alpha)/(R_1 + R_2 \cos \alpha)$$
$$= \{qR_2 \sin \alpha + cR_2 \cos \alpha - z(R_1 + R_2 \cos \alpha)\}/R_2 \sin \alpha,$$
$$\text{and} \quad -xR_2 \sin \alpha + y(R_1 + R_2 \cos \alpha) = 0.$$

The pitch $= \Sigma LX/\Sigma X^2$. Substitute and simplify, noting that

$$R_1/(x \sin \alpha - y \cos \alpha) = R_2/y = (R_1 + R_2 \cos \alpha)/x \sin \alpha$$

and $\tan \theta = y/x$.

The line of action of R_2 might be $y = x \tan \alpha$, $z = -c$.

426. Two wrenches, of pitches p and q and force intensities P and Q respectively, are equivalent to a wrench of pitch r. If ϖ is the pitch of the wrench which is equivalent to P and Q, prove that

$$pd_{qr} + qd_{rp} + rd_{pq} = \varpi \left\{ \frac{(p-q)^2}{d_{pq}} + d_{pq} \right\},$$

where d_{mn} denotes the shortest distance between the axes of wrenches of pitches m and n.

Let R be the resultant force, and put $d_{pq} = c$, $d_{rp} = -x$, $d_{qr} = x - c$. Prove that $R^2 = P^2 + Q^2 + 2PQ \cos \alpha$, $\varpi R^2 = cPQ \sin \alpha$,

$$rR^2 = pP^2 + qQ^2 + \{(p+q) \cos \alpha + c \sin \alpha\} PQ,$$
$$xR^2 = (p-q) PQ \sin \alpha + cQ(Q + P \cos \alpha),$$

and substitute in $\{p(x-c) - qx + rc\} R^2$.

427. A rigid body, acted on by wrenches, of respective force-intensities P_1, P_2, on screws of pitches p_1, p_2, receives a small twist $d\omega$ on the screw p_1; prove that the work done by the forces is

$$(2P_1p_1 + P_2\varpi_{12})d\omega,$$

where
$$\varpi_{12} = (p_1 + p_2) \cos \theta - h \sin \theta,$$

h being the shortest distance between the axes of the screws and θ the angle between them.

Hence, or otherwise, show that, if R is the force intensity and p the pitch of the resultant of any number of wrenches,

$$R^2 p = P_1^2 p_1 + P_2^2 p_2 + \dots + P_1 P_2 \varpi_{12} + \dots.$$

If a rigid body is acted upon by a wrench about a screw α, we can find in the following manner the work done by giving the body a small twist about another screw β.

Let AP and Ox (Fig. 238) represent the screws α and β, respectively,

their pitches being p_α and p_β. Let the force in the wrench be P, and the angle of rotation about β be $d\omega$, while θ is the angle (§ 216) between the screws.

Replace the force P and the couple $P.p_\alpha$ by their components at the point A parallel and perpendicular to Ox. The components of the couple are $Pp_\alpha \cos\theta$ and $Pp_\alpha \sin\theta$; and these we may suppose transferred to the point O. The components of the force are $P\cos\theta$ and $P\sin\theta$. Transfer these to O, introducing (§ 202) the couples whose axes along Oy and Ox are $Ph\cos\theta$ and $-Ph\sin\theta$, where $h = OA =$ shortest distance between the screws. Hence the given wrench is replaced by a force $P\cos\theta$ acting along Ox, a force $P\sin\theta$ acting along Oy, a couple $Pp_\alpha\cos\theta - Ph\sin\theta$ whose axis is along Ox, and a couple $Pp_\alpha\sin\theta + Ph\cos\theta$ whose axis is along Oy. For the displacement of translation $p_\beta d\omega$ along Ox the only work done is $P\cos\theta \times p_\beta d\omega$, which is due to the first component force; and for the rotation $d\omega$ round Ox the only work done (§ 201) is

$$(Pp_\alpha\cos\theta - Ph\sin\theta)\,d\omega,$$

which is due to the first component couple. Hence the whole work done is

$$P[(p_\alpha + p_\beta)\cos\theta - h\sin\theta]\,d\omega. \qquad (\alpha)$$

The expression in brackets is called the *virtual coefficient* of the two given screws.

This virtual coefficient may be denoted by $\varpi_{\alpha\beta}$. Then $\varpi_{\alpha\alpha} = 2p_\alpha$, $\because \theta = 0$.

For the second part we use the fact that the work done by the wrench (R, Rp) in twisting a body about any screw through any angle is equal to the sum of the works done by the component wrenches in the same twist.

Thus $2Rp = P_1\varpi_{01} + P_2\varpi_{02} + \dots$, where $\varpi_{0n} \equiv (p + p_n)\cos\theta - h\sin\theta$. Now consider a twist about the screw (1); then

$$R\varpi_{01} = 2R_1p_1 + R_2\varpi_{12} + R_3\varpi_{13} + \dots,$$

and similarly for ϖ_{02}, &c. Elimination of $\varpi_{01}, \varpi_{02}, \dots$, gives the result.

428. Prove that, if four wrenches, all of the same pitch, form a system in equilibrium, then the four forces and the four couples each separately form a system in equilibrium.

429. Prove that, if the axis of the wrench which is the resultant of wrenches on five given screws passes through a fixed point, it must always lie in a fixed plane.

430. Find the locus of the axis of a wrench which is the resultant of wrenches on two given screws.

Prove that, if the axis of the wrench which is the resultant of wrenches on five given screws lies in a fixed plane, it must pass through a fixed point.

431. If ω is the angle, d the shortest distance, between two screws of pitch a and b respectively, and the conic is described whose equation referred to axes parallel to those screws is $ax^2 + \varpi xy + by^2 = 1$,

where $\varpi = (a+b)\cos\omega - d\sin\omega$; prove that the pitch of the resultant of wrenches on the given screws varies inversely as the square of that radius vector of the conic to which its axis is parallel.

What is the meaning of the expression ϖ?

See § 219 and Ex. 427 (above).

432. Three forces and one couple act on a solid body; under what circumstances can they produce equilibrium?

433. A, B, C, D, E are five points not all in one plane. Prove that five forces acting along AB, BC, CD, DE, and EA cannot be in equilibrium.

434. O, O' are fixed points on two given lines, and the lines of action of five forces, whose components in a direction perpendicular to the two given lines are Z_1, Z_2, ..., intersect these lines in points P_1, P_2, ..., P_1', P_2', ... respectively. Show that if the forces are in equilibrium $\Sigma Z \,.\, OP \,.\, O'P' = 0$; and, conversely, that if this relation is true for all positions of O and O', the forces are in equilibrium.

435. Each force of a system is turned about an axis perpendicular to the Poinsot axis through a given angle, the points of application remaining fixed. Prove that the locus of new Poinsot axes for different values of the angle of rotation can have its equation put in the form $z^2(x^2+y^2) = k^2x^2$, where k is a constant.

436. A funicular polygon is in equilibrium under the action of forces acting at the corners, the components of the force acting at the r^{th} corner (x_r, y_r, z_r) being X_r, Y_r, Z_r. If a new polygon is constructed such that its sides are conjugate to those of the former with respect to $x^2+y^2+z^2+1=0$, prove that it will be in equilibrium under the action of forces such that the force at the r^{th} corner is (L_r, M_r, N_r), where

$$L_r = y_rZ_r - z_rY_r;\ M_r = z_rX_r - x_rZ_r;\ N_r = x_rY_r - y_rX_r.$$

437. A system of forces, whose directions in space are fixed, acts at assigned points of a rigid body. The body, being so placed that the forces have a single resultant, receives an infinitesimal displacement about an axis through a point on this resultant. Show that, if the axis of rotation lies in a certain plane, the forces will still have a single resultant.

438. Three points are taken on the axes of x, y, z respectively distant a from the origin. Forces P, Q, R act at those points in planes perpendicular to the axes and in directions making angles θ, ϕ, ψ with the axes of z, x, y respectively. Prove that if these forces reduce to a single resultant, the lines of action of the forces and the resultant will be generators of the surface

$$\Sigma\left[(\cos\theta-\sin\theta)\left\{(x^2-ax)\sin\phi\cos\psi+yz\sin\psi\cos\phi\right\}\right]$$
$$=(\cos\theta\cos\phi\cos\psi-\sin\theta\sin\phi\sin\psi)\left\{\Sigma yz-\Sigma ax+a^2\right\}.$$

We have $X = Q\cos\phi + R\sin\psi$, &c.; $L = a(Q\sin\phi - R\cos\psi)$, &c. Hence for a single resultant $\Sigma QR\cos(\phi+\psi) = 0$. In this case the

resultant acts along the central axis. Write down the equations of this axis, and prove that the resultant acts along a generator of the surface

$$\begin{vmatrix} z\sin\theta - y\cos\theta & -(y-a)\sin\phi & (z-a)\cos\psi \\ (x-a)\cos\theta & x\sin\phi - z\cos\phi & -(z-a)\sin\psi \\ -(x-a)\sin\theta & (y-a)\cos\phi & y\sin\psi - x\cos\psi \end{vmatrix} = 0.$$

From this determinant we observe that the lines of action of P, Q, R lie on the surface.

439. The components of the force acting upon a particle when it is at the point (x, y, z) are respectively

$$\alpha\frac{\partial\phi}{\partial x}, \quad \beta\frac{\partial\phi}{\partial y}, \quad \gamma\frac{\partial\phi}{\partial z},$$

when α, β, γ are positive quantities; prove that, if ϕ is a maximum when $x=x_0$, $y=y_0$, $z=z_0$, then (x_0, y_0, z_0) is a position of stable equilibrium.

440. The cross-section of a cylindrical body is the curve $\rho = f(s)$, where s, ρ are arc and radius of curvature, and the centre of gravity is on the normal at (ρ, s) at a distance h from its foot. Prove that, if the cylinder is placed on a rough horizontal plane with this normal vertical, it will not be in stable equilibrium (1) if $f(s) < h$, or (2) if $f(s) = h$ and $f'(s) \neq 0$; and examine cases when $f(s) = h$, $f'(s) = 0$, and $f''(s) \neq 0$.

441. A uniform hemisphere of mass m and radius a, having a mass m' attached to the centre of its circular base, rests in equilibrium with its vertex in contact with the top of a rough sphere of radius ka. Determine in terms of m and k the value m' that the equilibrium may be neutral to a first approximation, and investigate its true character.

442. A uniform elliptic cylinder, with flat ends at right angles to its axis, rests between two fixed smooth planes which make angles α, $\frac{1}{2}\pi+\alpha$ with the horizon towards the same side and intersect in a horizontal line to which the axis of the cylinder is parallel. Taking $4\alpha<\pi$, prove that, in order that there may be two positions of stable equilibrium, the eccentricity of the principal elliptic section must not be less than $\sec\alpha\sqrt{\cos 2\alpha}$.

Let the planes be those of yz and zx, the latter being inclined to the horizontal at the angle α; and let the xy-plane pass through the C. G. (C) of the cylinder. Let the normals at the points of contact intersect in $I(p, q)$. The equation of the cylinder is

$$x^2/p^2 + 2\lambda xy/pq + y^2/q^2 - 2x/p - 2y/q + 1 = 0$$

(where $\lambda^2 < 1$) and C is $\{p/(1+\lambda), q/(1+\lambda)\}$. If, then, $\lambda = 0$, the points C and I coincide, and the major axis of the cylinder is parallel to one of the planes. If these are both stable positions, a slight displacement must in each case raise the C. G. Hence, if $2a$, $2b$ are the

major and minor axes, $p = a+t$, $q = b-u$, where t, u are small quantities of the first order, both the inequalities

$$(p\cos\alpha+q\sin\alpha)/(1+\lambda) > a\cos\alpha+b\sin\alpha,$$

and

$$(p\sin\alpha+q\cos\alpha)/(1+\lambda) > a\sin\alpha+b\cos\alpha$$

must hold. If the second is satisfied, the first will be. Now

$$p^2+q^2 = (a^2+b^2)(1+\lambda)^2;\ \therefore at = bu+(a^2+b^2)\lambda,$$

small quantities of the second order being neglected. Also

$$pq = ab\sqrt{(1+\lambda)^3(1-\lambda)},$$

since the area of the ellipse is unaltered. $\therefore ab+bt-au = ab(1+2\lambda)$ and $bt-au = 2\lambda ab$. Hence

$$(a+t)\sin\alpha+(b-u)\cos\alpha-a(1+\lambda)\sin\alpha-b(1+\lambda)\cos\alpha$$
$$= (u+\lambda b)(a\sin\alpha-b\cos\alpha)/b.$$

$\therefore a\sin\alpha > b\cos\alpha$, which gives the required result.

If, on the other hand, $\lambda \neq 0$, then CI, which passes through the origin, must be vertical. This involves the equality $p\sin\alpha = q\cos\alpha$, and the positions of unstable equilibrium can be determined, e.g. by finding the inclination of the major axis to the axis of x. It will be seen that $a\sin\alpha$ must be greater than $b\cos\alpha$ in order that this angle may be real.

443. A solid in the form of a portion of a paraboloid of revolution is placed with its vertex resting upon a horizontal plane, and is so loaded that the centre of curvature at the vertex coincides with the C. G. Determine whether the equilibrium is stable or unstable.

444. Show that a heavy uniform solid cylinder, whose cross-section is the cardioid $r = a(1-\cos\theta)$, will rest in equilibrium on a perfectly rough plane inclined to the horizon at an angle $\tan^{-1}\frac{1}{11}$, with the generator corresponding to $\theta = \pi/2$ horizontal and in contact with the plane, and show that the equilibrium is unstable.

The C. G. is distant $5a/6$ from the cusp and the normal at the point of contact makes an angle $\frac{1}{4}\pi$ with the radius vector.

445. A uniform rod of length $2a$ rests in a vertical position with its lower end on a smooth surface of revolution, and passes through a small smooth ring which is fixed at a distance h from this end. In this position the rod lies along the axis of the surface; the concavity of the surface is upwards, and ρ is the radius of curvature of the meridian curve at the point of contact. Prove that the position is stable if $h^2 > a\rho$.

446. A segment of the uniform spheroid $x^2+2y^2+2z^2=a^2$ rests with the vertex $(a, 0, 0)$ in contact with a fixed rough horizontal plane. Determine the length of the axis of the segment for which the equilibrium (for a slight rolling displacement) is apparently neutral, and show that it is really stable.

Result. $\frac{1}{3}a(5-\sqrt{7})$. Taking $a^2b^2 = \rho(a^2\cos^2\psi+b^2\sin^2\psi)^{\frac{3}{2}}$, prove that $(d/ds)^2\rho^{-1}$ is negative when $\psi = 0$.

447. A heavy circular cylinder of radius a rests in equilibrium inside a fixed perfectly rough cylindrical surface, the generating lines of both surfaces being horizontal. If the section of the fixed surface has the form of the catenary $s = c \tan \psi$, where s, ψ are measured from the lowest point, and if the circular cylinder touches it along its lowest generator, and has its centre of gravity at a height $4a/3$ vertically above that generator, show that the equilibrium is stable unless c is greater than $4a$.

When the tangent at the point of contact is inclined at a small angle ψ to the horizon, $s = c(\psi + \frac{1}{3}\psi^3)$ and the C. G. of the cylinder has risen through a vertical distance

$$c(\sec\psi - 1) + a \sin(s/a)\sin(s/a - \psi)$$
$$+ (\tfrac{1}{3}a + a\cos s/a)\cos(s/a - \psi) - 4a/3$$

which reduces to $(4a - c)(c - a)\psi^2/6a +$ higher powers. This is > 0, if $a < c < 4a$. If $c = a$ or $4a$, the coefficient of ψ^4 is > 0 and the equilibrium is stable.

448. A circular cylinder of radius a is fixed and a perfectly rough cylinder of radius b is placed in equilibrium on it; the axes of the cylinders are horizontal and inclined to one another at an angle α; show that for displacements between the planes

$$x = 0 \text{ and } (b + a\cos 2a)\, x = ay \sin 2\alpha$$

the equilibrium is stable, the axis of z being the common normal and the axis of y the highest generator of the fixed cylinder.

449. A homogeneous ellipsoid, whose semi-axes are a, b, c $(a > b > c)$, is placed upon an equal and similarly situated ellipsoid, so that their least axes are in the same vertical line; prove that, if $a^{-2} + b^{-2} = c^{-2}$, the equilibrium is stable, unstable, or neutral, according as the displacement of the centre of the first ellipsoid takes place in the plane of a, c or of b, c, or that bisecting the angle between those planes.

Let $c = a \sin\alpha = b\cos\alpha$, where $0 < \alpha < \frac{1}{4}\pi$. The height of the C. G. above the point of contact is c. The radii of curvature of the first two sections are a^2/c, b^2/c, i.e. $c \operatorname{cosec}^2\alpha$, $c\sec^2\alpha$, respectively. The former $> 2h$, $\therefore$ the equilibrium is stable; the latter $< 2c$, $\therefore$ unstable equilibrium. The third radius of curvature is the H. M. of the other two and can be shown to be equal to $2c$, $\therefore$ neutral equilibrium.

450. A body rests in equilibrium under the action of gravity with one point in contact with another fixed rough body, and the equations of the surfaces of the two bodies referred to the point of contact as origin, and the common normal as axis of z, are

$$2z = ax^2 + 2hxy + by^2 + \dots, \quad -2z = a'x^2 + 2h'xy + b'y^2 + \dots.$$

Show that in discussing the stability of the system we may replace the lower body by the tangent plane at the point of contact, and the upper body by a body the equation of whose surface is

$$2z = (a + a')x^2 + 2(h + h')xy + (b + b')y^2 + \dots,$$

and whose C. G. is at the same point as that of the body replaced.

A homogeneous ellipsoid whose axes are a, b, c ($a > b > c$ and $2b^2 > a^2 + c^2$) rests in equilibrium with an umbilic in contact with the rough surface of a fixed sphere. Determine the least value of the radius of the latter which is compatible with the equilibrium being thoroughly stable.

451. The density within the surface expressed in polar co-ordinates by the equation $r = a + b\cos\theta$, where b is small compared with a, is given by the formula $\rho(1+k)$, where ρ is constant and

$$k(s^2a^2 - 5) = (s^2r^2 - 5)bs^2r\cos\theta \exp.\{\tfrac{1}{2}s^2(a^2 - r^2)\},$$

s being a constant such that b/a is small compared with $s^2a^2 - 5$. Neglecting b^2, prove that the C. G. is at a distance from the origin equal to $5b/(s^2a^2 - 5)$, and that one of the equipotential surfaces is a sphere of radius a with its centre at the C. G.

(N.B.—*In many of the following examples the gravitation constant, γ, has been omitted, i.e. assumed to be unity.*)

452. The rim of a uniform plate is the curve

$$z = 0,\ x^2/a^2 + y^2/b^2 = 1\ ;$$

find the potential of the plate at any point of the hyperbola

$$y = 0,\ x^2/(a^2 - b^2) - z^2/b^2 = 1.$$

Slight reflection will show that we cannot deduce the desired result from equation (19) of § 344 by assigning an infinitesimal value to c, for the superficial density would not be uniform. In fact, at a point $(x, y, 0)$ it would be a multiple of the thickness of the (infinitesimal) stratum at that point, i.e. of $1 - x^2/a^2 - y^2/b^2$. If, on the other hand, we obtain for an external point the potential of a heterogeneous ellipsoid whose strata of equal density are similar to, and concentric with, the ellipsoid, we can without difficulty deduce the potential of a uniform plate. (The student may consult with advantage Routh, *Statics*, vol. ii, §§ 239, 251.)

The potential of the plate, supposed of uniform superficial density equal to unity, at the point (x, y, z) is

$$V = \gamma\iint d\xi d\eta/\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}$$

taken over the area of the ellipse $\xi^2/a^2 + \eta^2/b^2 = 1$. By the substitutions $\xi = ma\cos u$, $\eta = mb\sin u$, this can be transformed into

$$V = ab\gamma\iint m\,dm\,du/\sqrt{(x - ma\cos u)^2 + (y - mb\sin u)^2 + z^2},$$

the limits of m being 0 and 1, and those of u being 0 and 2π. Now the substitution $\tan\frac{1}{2}u = (p + q\tan\frac{1}{2}\phi)/(1 + r\tan\frac{1}{2}\phi)$, where p, q, r are suitably chosen, reduces this integral to a form analogous to that for the potential of a homogeneous ellipsoid at an external point; but, as the corresponding values of $\cos u$ and $\sin u$ are of the forms

$$(\alpha\cos\phi + \alpha'\sin\phi + \alpha'')/(\gamma\cos\phi + \gamma'\sin\phi + \gamma'')$$

and $$(\beta\cos\phi + \beta'\sin\phi + \beta'')/(\gamma\cos\phi + \gamma'\sin\phi + \gamma'')$$

respectively, it is convenient to make use of these. From the identities $\cos^2 u + \sin^2 u = \cos^2\phi + \sin^2\phi = 1$ we find that

$$\alpha^2+\beta^2-\gamma^2 = \alpha'^2+\beta'^2-\gamma'^2 = -\alpha''^2-\beta''^2+\gamma''^2 = h^2 \text{ say,}$$

and $\alpha\alpha'+\beta\beta'-\gamma\gamma' = \alpha'\alpha''+\beta'\beta''-\gamma'\gamma'' = \alpha''\alpha+\beta''\beta-\gamma''\gamma = 0.$

Since, however, we may divide by h every term of the numerators and denominators of $\cos u$ and $\sin u$, we introduce no limitation by putting $h = 1$. (N.B.—Here γ is not the gravitation constant.)

Now, substituting for $\cos u$ and $\sin u$, we obtain

$$(x-ma\cos u)^2+(y-mb\sin u)^2+z^2$$
$$= (G\cos^2\phi+G'\sin^2\phi+G'')/(\gamma+\gamma'\cos\phi+\gamma''\sin\phi)^2,$$

where $G\alpha^2+G'\alpha'^2+G''\alpha''^2 = m^2a^2$, $G\beta^2+G'\beta'^2+G''\beta''^2 = m^2b^2$,

$$G\gamma^2+G'\gamma'^2+G''\gamma''^2 = x^2+y^2+z^2,\ -G\beta\gamma-G'\beta'\gamma'-G''\beta''\gamma'' = -mby,$$
$$-G\gamma\alpha-G'\gamma'\alpha'-G''\gamma''\alpha'' = -max,\ G\alpha\beta+G'\alpha'\beta'+G''\alpha''\beta'' = 0.$$

By virtue of the preceding identical relations between the constants these conditions lead to

$$(G-m^2a^2)\,\alpha+max\gamma = 0,\ (G-m^2b^2)\,\beta+mby\gamma = 0$$
$$-max\alpha-mby\beta+(G+x^2+y^2+z^2)\,\gamma = 0.$$

In order that these equations may be consistent, $-G$ must satisfy a cubic equation in θ, viz.: $x^2/(m^2a^2+\theta)+y^2/(m^2b^2+\theta)+z^2/\theta = 1$. Similarly, $-G'$, $+G''$ can be shown to be roots of this equation, which, if $a > b$, has one positive and two negative roots. Hence G'' is the positive root.

Since α, α', $i\alpha''$; β, β', $i\beta''$; $-i\gamma$, $-i\gamma'$, γ'' satisfy the conditions for an orthogonal system of direction cosines, we can easily prove that $du = d\phi/(\gamma\cos\phi+\gamma'\sin\phi+\gamma'')$. Accordingly

$$V = 4ab\gamma\int m\,dm\int d\phi/(G\cos^2\phi+G'\sin^2\phi+G'')^{\frac{1}{2}},$$

the limits of m being 0 and 1, and 0 and $\frac{1}{2}\pi$ those of ϕ.

Now write t for $G''\operatorname{cosec}^2\phi+G\cot^2\phi$ and simplify. We obtain

$$V = 2ab\gamma\int_0^1 m\,dm\int_\theta^\infty dt$$
$$/\sqrt{t\,(t+m^2a^2)\,(t+m^2b^2)\,\{1-x^2/(t+m^2a^2)-y^2/(t+m^2b^2)-z^2/t\}},$$

where θ is the positive root of the equation

$$x^2/(\theta+m^2a^2)+y^2/(\theta+m^2b^2)+z^2/\theta = 1.$$

The value of the integral with respect to t will not be altered if we write m^2t for t and $m^2\theta$ for θ. This modification gives for the expression under the radical

$$t\,(t+a^2)\,(t+b^2)\,\{m^2-x^2/(t+a^2)-y^2/(t+b^2)-z^2/t\},$$

while θ is now the positive root of the equation

$$x^2/(\theta+a^2)+y^2/(\theta+b^2)+z^2/\theta = m^2,$$

and is obviously a function of m. We may, however, invert the order of integration and obtain

$$V = 2ab\gamma\left[\int_\theta^\infty dt\sqrt{m^2 - x^2/(t+a^2) - y^2/(t+b^2) - z^2/t}\Big/\sqrt{t(t+a^2)(t+b^2)}\right]_0^1,$$

the legitimacy of the inversion being apparent if we differentiate this last integral with respect to m. The term depending on the limit θ is, namely,

$$-\sqrt{m^2 - x^2/(\theta+a^2) - y^2/(\theta+b^2) - z^2/\theta}\;.\;d\theta/dm\sqrt{\theta(\theta+a^2)(\theta+b^2)},$$

which vanishes in virtue of the equation which defines θ. Completing the integration, and noting that, when $m = 0$, $\theta = \infty$, we finally obtain

$$V = 2ab\gamma\int_\theta^\infty dt\sqrt{1 - x^2/(t+a^2) - y^2/(t+b^2) - z^2/t}\Big/\sqrt{t(t+a^2)(t+b^2)},$$

where θ is defined as above.

In the present case $y = 0$, $x^2/(a^2 - b^2) - z^2/b^2 = 1$. Using these relations we can deduce that

$$1 - x^2/(a^2+t) - y^2/(b^2+t) - z^2/t = (b^2+t)(t - z^2a^2/b^2)/t(a^2+t)$$

and that $\theta = z^2a^2/b^2$. Hence

$$V = 2ab\gamma\int_\theta^\infty dt\sqrt{t-\theta}/t(a^2+t) = 2\pi\gamma(\sqrt{z^2+b^2} - z),$$

which is independent of a.

In a Paper by Cayley (*Proceedings of the London Mathematical Society*, vol. vi) an equivalent result is obtained by another method. As this is frequently applicable to problems involving plane attracting areas, an outline may be given here.

The hyperbola of the question is the focal hyperbola of the quadric $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ when $c \longrightarrow 0$. Hence any point P on the hyperbola determines with the ellipse a right cone of which the axis is the tangent to the hyperbola at P. The potential of the ellipse at P is given by $V = \gamma\int r^2 d\omega/z$, where $d\omega$ is the solid angle subtended at P by an element of the ellipse, and r the distance of the element from P.

Now take P as origin, the tangent as the axis of z', and the axis of y' in the plane drawn through P parallel to the plate. If then α is the angle between the axes of z and z', the ellipse lies in the plane $x'\sin\alpha + z'\cos\alpha = z$. Substitute $r\sin\theta\cos\phi$ for x', $r\sin\theta\sin\phi$ for y', $r\cos\theta$ for z', and $\sin\theta d\theta d\phi$ for $d\omega$. Then we find r from the equation $r(\sin\alpha\cos\phi\sin\theta + \cos\alpha\cos\theta) = z$, and the integral for V has to be taken over the whole spherical aperture of the cone, i. e. from $\theta = 0$ to $\theta = \Theta$, and from $\phi = 0$ to $\phi = 2\pi$. After somewhat laborious calculations it is found that

$$V = 2\pi\gamma z(\cos\alpha/\sqrt{\cos^2\alpha - \sin^2\Theta} - 1) = 2\pi\gamma(\sqrt{z^2+b^2} - z),$$

as before.

453. Prove that the potential of a circular ring of radius a, density ρ, and small transverse section $\delta\omega$ at an external point (x, y) in the plane of the ring is

$$2\gamma\rho a\delta\omega\int_{\mu}^{\infty} d\theta/\sqrt{\theta(\theta-\mu)(\theta+a^2)},$$

where $\mu = x^2+y^2-a^2$.

(Prove that

$$V = 4\gamma\rho a\int_0^{\frac{1}{2}\pi} d\phi/\sqrt{r^2-a^2\sin^2\phi},$$

where $r^2 = x^2+y^2$, and then put $r^2\cot^2\phi = \theta-\mu$.)

454. Of all the points within a triangle and in its plane, that at which the potential of the perimeter is least is the centre of the inscribed circle.

455. Show by the use of the potential that the mutual attraction along their shortest distance of two infinite uniform lines at right angles is 2π, the mutual attraction of unit lengths at unit distance being unity.

456. Prove that the potential of the six faces of a cube at the centre of the cube is $M\gamma\{6\log(2+\sqrt{3})-\pi\}/a$, where $2a$ is the length of a side and M the mass of a face, supposed infinitely thin.

This potential may be deduced in the following manner from the result of § 325, Ex. 3. The attraction of a square plate, of side $2a$ and mass M, at a point on the perpendicular at the centre, is $\gamma Ma^{-2}\sin^{-1}\{a^2/(h^2+a^2)\}$, where h is the distance of the point from the plate, and this expression $= -dV'/dh$, if V' is the potential of the plate. Since $V'=0$, when $h=\infty$, we have at the centre of the cube

$V' = \gamma Ma^{-2}\int_a^{\infty} dh\sin^{-1}\{a^2/(h^2+a^2)\}$. Put $h = ax$, and integrate by parts. We obtain

$$aV'/\gamma M = \Big[x\sin^{-1}1/(1+x^2)\Big]_1^{\infty} + 2\int_1^{\infty} x\,dx/(1+x^2)\sqrt{2+x^2},$$

or, if $(1+x^2)z = 1$,

$$aV'/\gamma M = -\pi/6+\int_0^{\frac{1}{2}} dz/\sqrt{z+z^2} = -\pi/6+\log(2+\sqrt{3}).$$

Hence for the six faces

$$V = 6V' = M\gamma\{6\log(2+\sqrt{3})-\pi\}/a = kM/a, \text{ say.}$$

We may deduce the potential of the whole cube in the following manner. The potential of the six faces of a cube of edge $2x$ is $4x^2\rho dx\,.\,k/x$. Integrate this from $x=0$ to $x=a$, and let $M = 8\rho a^3$ be the mass of the cube. The result is

$$Mk/4a = M\gamma\{6\log(2+\sqrt{3})-\pi\}/4a.$$

It is expedient to exhibit another method, which is of considerable importance for the treatment of this type of problem.

Let O be the centre of one face, P any other point in it, C the centre of the cube, $R = OP$, $r = CP$, ϕ the angle between OP and a side of the square, σ the density. Then $r^2 = R^2 + a^2$, and $V = 6\sigma\gamma\int\int rdrd\phi/r$, taken over the face. If OP intersects in Q the first edge which it meets, the limits of r are $CO\,(=a)$ and $CP\,(= a\sqrt{\sec^2\phi + 1})$. Also, the limits of ϕ being 0 and 2π, it will be seen that we may integrate from 0 to $\frac{1}{4}\pi$, and then multiply the result by 8, thus dividing the square face into eight isosceles right-angled triangles. It can further be proved [by putting $\tan\phi = t$ and $(t^2+1)\,y^2 = t^2+2$], that

$$\int_0^{\frac{1}{4}\pi} \sqrt{\sec^2\phi + 1}\,d\phi = \log\{(1+\sqrt{3})/\sqrt{2}\} + \pi/6.$$

Hence we obtain the result kM/a as above, where $M = 4\sigma a^2$.

In order to obtain the potential of the cube at C, let $d\omega$ be the solid angle subtended at that point by the element dS at P, where $CP = r$. Then $V = 6\gamma\int\int\int\rho r^2 dr d\omega/r = 3\gamma\rho\int\int r^2 d\omega$. Now the volume of the cone with vertex C and base dS can be expressed both in the form $\frac{1}{3}a dS$ and as $\frac{1}{3}r^3 d\omega$. Hence $V = 3a\rho\gamma\int\int dS/r$ taken over an external square face, and this, by what precedes, is equal to $3a\rho \,.\, 4a^2k/6a = Mk/4a$, if M is the mass of the cube.

By means of either of the preceding methods we may find the potential of the cube at one of its corners A. The second mode of procedure may be found instructive.

As before, $V = \frac{1}{2}\gamma\rho\int\int r^2 d\omega$ taken over the six faces. For the three faces which meet in $A d\omega = 0$, and for each of the others $\int\int r^2 d\omega$ is the same. $\therefore\ V = \frac{3}{2}\gamma\rho\int\int r^2 d\omega = \frac{3}{2}\gamma\rho\int\int 2a dS/r = 3\rho a \times$ the potential at A of the square face supposed of superficial density $= 1$. Now $\int\int dS/r = \int\int R dR d\phi/\sqrt{R^2+4a^2}$, where R, ϕ are the co-ordinates of a specimen element referred to the corner nearest to A as pole and to a side as initial line. We have again to use $\int_0^{\frac{1}{4}\pi}\sqrt{\sec^2\phi+1}\,d\phi$, and we finally obtain $\gamma M\ \log\{6\log(2+\sqrt{3}) - \pi\}/8a$, or half the value at C, the centre of the cube.

457. If two attracting systems are to have the same potential at every point outside a surface enclosing both of them, show that it is necessary but not sufficient that their centres of mass and principal axes should coincide, that their masses should be equal, and that the differences of their moments of inertia about any straight line should be constant.

Prove that the potentials of an ellipsoid and a rectangular parallelepiped of uniform densities at points similarly situated with respect to their principal axes and distant r from their centres, will only differ by quantities varying as $1/r^5$ and higher powers, if their axes are proportional and the ratio of their densities is

$$\frac{6}{\pi}\left(\sqrt{\frac{3}{5}}\right)^3.$$

458. Find the potential of a lamina in the form of an equilateral triangle, at a point on the line drawn through its centre of inertia normal to its plane, and show that the potential of an equilateral solid tetrahedron at its centre of inertia is

$$M\gamma\{6\log(\sqrt{3}+\sqrt{2})-\tfrac{1}{2}\sqrt{2}\,\pi\}/a,$$

where M is its mass and a the length of an edge.

The general question of the potentials of polygons and polyhedra at a point P is treated in detail in a Paper by Cayley (*Proceedings of the London Mathematical Society*, vol. vi). The method adopted depends upon the division of the polygon into right-angled triangles by means of lines drawn from the foot of the perpendicular dropped from P on the plane of the polygon.

If a is the side of the triangle ABC, M its mass, O its centre, P any point on the perpendicular to its plane at O, and α the angle OPA, it may be shown, by dividing the triangle into strips parallel to one side, that V, the potential at

$$P=\frac{2\gamma M\cot\alpha}{a}\left\{4\tan^{-1}(\sqrt{3}\cos\alpha)-\frac{4\pi}{3}+\tan\alpha\log\frac{2+\sqrt{3}\sin\alpha}{2-\sqrt{3}\sin\alpha}\right\}.$$

The following is an alternative method:

Let (R,ϕ) be the co-ordinates of any element Q of the triangle referred to O as pole; $r=(R^2+h^2)^{\frac{1}{2}}$, where r, h are the distances of P from Q and O respectively; σ the density, so that $4M=\sigma a^2\sqrt{3}$; $p\,(=a/2\sqrt{3})$ the perpendicular from O on a side. Then $RdR=rdr$, and $V=\gamma\int\int\sigma r\,dr\,d\phi/r=\gamma\sigma\int(r_1-h)\,d\phi$, where (R_1,ϕ) is the point, F, where OQ meets the side of ABC and $r_1{}^2=R_1{}^2+h^2$. Hence $V/\gamma=\int R_1{}^2d\phi/r_1-h\int(1-h/r_1)\,d\phi$. If dx is an element of the side at F, we have $R_1{}^2d\phi=p\,dx$; and, if ω is the solid angle subtended at P by the lamina, $d\omega=R_1dR_1d\phi\cos FPO/r_1{}^2=h\,dr_1d\phi/r_1{}^2$. $\therefore V/\gamma=\sigma\left[\int p\,dx/r_1-h\omega\right]=3V'\sigma-h\sigma\omega$, where V' is the potential at P of the side BC, supposed of unit linear density.

By § 331, Ex. 2,

$$\begin{aligned}V'&=\gamma\log\{(PA+PB+a)/(PA+PB-a)\}\\&=\gamma\log\{(2h\sec\alpha+a)/(2h\sec\alpha-a)\}\\&=\gamma\log\{(2/\sqrt{3}+\sin\alpha)/(2/\sqrt{3}-\sin\alpha)\}.\end{aligned}$$

In order to calculate the value of ω, describe, with centre P and unit radius, a sphere to cut PA, PB, PC, PO in A', B', C', O' respectively. Then $\omega=$ the area of the triangle $A'B'C'$, or, if the great circle $A'O'$ cuts $B'C'$ in D', $=6\Delta OB'D'=6(\angle O'B'D'+\angle B'O'D'-\tfrac{1}{2}\pi)$.

Now $\qquad \sin O'B'D'=\sin O'D'/\sin\alpha=1/(1+3\cos^2\alpha)^{\frac{1}{2}},$

or $\qquad \tan O'B'D'=1/\sqrt{3}\cos\alpha.$

Also $6\angle B'O'D'=2\pi$. $\therefore\ \omega=6\{\pi/3-\tan^{-1}(\sqrt{3}\cos\alpha)\}$: whence the result, say, $V=2\gamma Mf(\alpha)/a=\tfrac{1}{2}\gamma k\sigma af(\alpha)\sqrt{3}$.

In order to obtain the potential of the equilateral solid tetrahedron at its centre, G, we may use the method applied to the solid cube

in Ex. 456. This gives $V(G) = \frac{1}{2}\rho \times 4GO \times \frac{1}{2}\gamma\kappa a\sqrt{3}f(\alpha)$. Now $OG = \sqrt{(2/3)}\,a/4$; $\tan\alpha = AO/OG = 2\sqrt{2}$; $M = a^3\rho\sqrt{2}/12$.

459. Prove that the potential of a uniform hemisphere of radius a and density ρ at an external point on its axis distant z from its centre is

$$\tfrac{2}{3}\pi\gamma\rho\,[a^3/z \pm \{z^2 + \tfrac{3}{2}\,a^2 - (a^2+z^2)^{\frac{3}{2}}/z\}],$$

where the upper sign is to be taken when the point is nearer to the flat portion of the boundary, and the lower sign when it is nearer to the curved portion.

Prove also that the potential at an internal point on the axis is

$$\tfrac{2}{3}\pi\gamma\rho\,[-a^3/z - 2z^2 + \tfrac{3}{2}a^2 + (a^2+z^2)^{\frac{3}{2}}/z].$$

Let O be the centre, P a point on the axis beyond the curved portion, $OP = \xi > a$; Q a point in PO produced, $OQ = \eta$; R a point in OP, $OR = x < a$. Then, by considering a slice bounded by parallel planes drawn perpendicular to the axis, we obtain, for the potential at P,

$$V(P) = \int_0^a 2\pi\rho\gamma dx\,\{\sqrt{(\xi-x)^2 + a^2 - x^2} - (\xi - x)\}$$
$$= 2\pi\gamma\rho\,\{a^3 + (a^2+\xi^2)^{\frac{3}{2}} - \xi^3 - \tfrac{3}{2}a^2\xi\}/3\xi.$$

Similarly,

$$V(Q) = \int_0^a 2\pi\rho\gamma dx\,\{\sqrt{(\eta+x)^2 + a^2 - x^2} - (\eta + x)\}$$
$$= 2\pi\gamma\rho\,\{a^3 - (a^2+\eta^2)^{\frac{3}{2}} + \eta^3 + \tfrac{3}{2}a^2\eta\}/3\eta.$$

Now the potential of the complete sphere at Q is $2\pi\gamma\rho\,(a^2 - \frac{1}{3}\eta^2)$, if $\eta < a$. Subtract $V(Q)$.

460. If the potential of a body, symmetrical with regard to the plane of xy, and to the axis of z, at any external point on that axis is $f(z^2)$, show that the potential at any external point in the plane of xy, at a distance ρ from the axis, is given by

$$\frac{1}{\pi}\int_0^{\rho^2} \frac{f(-t)\,dt}{\sqrt{t(\rho^2 - t)}},$$

provided this integral has precise finite meaning.

461. Prove that the potential of a uniform circular ring of mass m at a point P is equal to m/r, where r is the arithmetico-geometrical mean between the greatest and least distances of P from the ring. (The arithmetico-geometrical mean between two positive quantities p and q is the common value of

$$\operatorname*{Lt}_{n=\infty} p_n \quad\text{or}\quad \operatorname*{Lt}_{n=\infty} q_n,$$

where p_n and q_n are formed as follows:

$$p_1 = \tfrac{1}{2}(p+q); \quad q_1 = \sqrt{pq}; \quad \ldots\ldots$$
$$p_{n+1} = \tfrac{1}{2}(p_n + q_n); \quad q_{n+1} = \sqrt{p_n q_n}.)$$

If the point P is very close to the ring, prove that the value of the potential is approximately $(m/\pi a)\log(8a/p)$, where a is the radius

of the ring and p is the shortest distance of the point from the circumference.

For a full treatment of this question the student may refer to Poincaré, *Théorie du potentiel newtonien*, §§ 17, 18, 59. The following is a sketch of his method of treatment (slightly modified).

Let Q be the projection of P on the plane of the ring; C the centre and a the radius of the ring; A, B the points in which QC cuts the ring; $PA=p$, $PB(>PA)=q$; $CQ=\rho$; $PQ=z$; $m=2\pi a\mu$. Then, if F is any point on the ring, $\angle FCA=\theta$, we have $V=\mu\int a\,d\theta/r$, where $r=FP$. It can be shown that $r^2=p^2\cos^2\frac{1}{2}\theta+q^2\sin^2\frac{1}{2}\theta$, and hence that $V/m=\int_0^{2\pi}d\theta/2\pi\sqrt{p^2\cos^2\frac{1}{2}\theta+q^2\sin\frac{1}{2}\theta}$. Put $\theta=2\theta_1$.

Then $2\pi V/m=\int_0^{2\pi}d\theta_1/\sqrt{p^2\cos^2\theta_1+q^2\sin^2\theta_1}$.

Now consider the integral

$$I=\int_0^{\pi}d\phi/\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}\quad\text{or}\quad\int_0^{\pi}d\phi/a\sqrt{1-\kappa^2\sin^2\phi},$$

where $\kappa^2=1-b^2/a^2$. Put $\kappa\sin\phi=\sin(2\psi-\phi)$, which is Landen's Transformation. Then

$$I=\int_0^{\frac{1}{2}\pi}2d\psi/a(1+\kappa)\sqrt{1-4\kappa\sin^2\psi/(1+\kappa)^2}$$

$$=\int_0^{\pi}d\psi/\sqrt{a^2(1+\kappa)^2\cos^2\psi+b^2(1-\kappa)^2\sin^2\psi}$$

$$=\int_0^{\pi}d\psi/\sqrt{a'^2\cos^2\psi+b'^2\sin^2\psi},$$

where $a=\frac{1}{2}(a'+b')$, $b=\sqrt{a'b'}$. If, then, $2\pi V/m\equiv f(p,q)$, we see that $f(p,q)=f(p_1,q_1)=\dots=f(p_n,q_n)$ when $n\to\infty$. Since $f(p,q)=f(q,p)$, we may assume $q>p$. Hence $q-p>q_1-p_1>q_2-p_2$, &c., and, since $p_1>p$, we have $q_1-p_1<q_1-p<\frac{1}{2}(q-p)$, and so on. Finally, $q_n-p_n<(q-p)/2^n$, which $\to 0$ as $n\to 0$. Therefore, in the limit, $q_n=p_n=r$, as defined in the question. Hence

$$2\pi V/m=\int_0^{2\pi}d\chi/r\quad\text{or}\quad V=m/r.$$

When P is very close to the ring, $q\to(p+2a)$, and very approximately

$$\pi(p+2a)V/2m=J\ (\text{say})=\int_0^{\frac{1}{2}\pi}d\theta/\sqrt{1-\kappa^2\sin^2\theta},$$

where $(p+2a)^2=4a\kappa^2(p+a)$, i.e. $p\to 0$ and $\kappa\to 1$.

In order to obtain the limit of J, divide the range from 0 to $\frac{1}{2}\pi$ into two parts, viz. from 0 to $\frac{1}{2}\pi-\alpha$ and from $\frac{1}{2}\pi-\alpha$ to $\frac{1}{2}\pi$, where α is very small, but α/κ', where $\kappa^2+\kappa'^2=1$, very large. Put $\frac{1}{2}\pi-\alpha=\beta$. Then throughout the first portion, i.e. from 0 to β,

$1-\kappa^2\sin^2\theta$, which $=\kappa'^2\sin^2\theta+\cos^2\theta$, is very nearly $=\cos^2\theta$. Hence

$$\int_0^\beta d\theta/\sqrt{1-\kappa^2\sin^2\theta}=\int_0^\beta d\theta\sec\theta=\log\tan\tfrac{1}{2}(\pi-\beta)=\log(2/\beta).$$

In the second portion write $\frac{1}{2}\pi-\phi$ for θ: then ϕ is very small throughout, and therefore we may put ϕ for $\sin\phi$ and 1 for $\cos\phi$. This gives

$$\int_\theta^\beta d\phi/\sqrt{\kappa'^2+\kappa^2\phi^2}=(1/\kappa)\log\{(\kappa\beta+\sqrt{\kappa'^2+\kappa^2\beta^2})/\kappa'\}$$
$$=(1/\kappa)\log(2\kappa\beta'/\kappa'),$$

if $\kappa'/\kappa\alpha$ is neglected, $=\log(2\beta/\kappa')$ very nearly, since $\kappa\longrightarrow 1$. Hence, approximately, $J=\log(2/\beta)+\log(2\beta/\kappa')=\log(4/\kappa')$. Now, here, when $p\longrightarrow 0$, $\kappa'^2=1-\kappa^2=p^2/(p+2a)^2\longrightarrow p^2/4a^2$, and we have $\pi aV/m\longrightarrow\log(8a/p)$.

An alternative method of finding the limit of J, when $p\longrightarrow 0$, is given in Routh, *Statics*, vol. ii, § 191.

462. An infinitely long cylinder, whose cross-section is an oval of Cassini ($r_1 r_2=a^2$), is coated with attracting matter whose surface density at any point is proportional to its distance from the axis of the cylinder. Prove that the external level surfaces are cylinders whose cross-sections are confocal Cassinians.

463. The potential of a solid homogeneous ellipsoid of revolution about a transverse axis at a focus is half as great again as that of the same mass distributed along the perimeter of the equator of the ellipsoid.

$$V=\rho\gamma\textstyle\iint 2\pi\gamma r^{-1}dxdy,\ \text{integrated over the semi-ellipse,}$$
$$=2\pi\rho\gamma I,\ \text{where}\ I=\textstyle\iint y\,dxdy/\sqrt{(x-c)^2+y^2},$$
$$=\int_{-c}^{a}[(a+ex)-(x+c)]\,dx+\int_{-a}^{-c}[(a+ex)+(x+c)]\,dx$$
$$=a^2-c^2=b^2.$$

The mass $m=\rho\iint 2\pi dxdy$ over the same area $=2\pi\rho ab^2/3$. $\therefore\ V=3m/2a$.

464. An ellipsoid is formed by the revolution of an ellipse of eccentricity e about its minor axis. Prove that, if e^4 may be neglected, the potential at an external point whose distance is R and latitude λ is $M/R+Ma^2e^2(1-3\sin^2\lambda)/10R^3$, where a is the equatorial radius of the ellipsoid and $R>a$. See § 354.

465. Matter attracting according to the law of nature is distributed along the circumference of an ellipse in such a way that the density at any point is inversely proportional to the length of the diameter parallel to the tangent at the point. Show that the chord of contact of tangents drawn to the ellipse from any external point divides the ellipse into two arcs such that the potentials at the point due to each arc are the same.

466. The density at each point inside a given sphere varies inversely as the fifth power of the distance from a given point, whose distance from the centre of the sphere is four times the radius. Prove that the point within the sphere at which the potential is a maximum divides the diameter drawn through it in the ratio 3 : 10. See § 333.

467. A thick shell is bounded by concentric and coaxial ellipsoids of semi-axes a, b, c; ma, mb, mc; where $m<1$. Prove that its potential is the same at all points of its empty interior; and find that potential.

If V is the potential and M the mass, prove that

$$c = a \operatorname{cn}\left(\frac{2V}{3M} \cdot \frac{1-m^3}{1-m^2} \sqrt{a^2-c^2}\right), \text{ mod. } \sqrt{\frac{a^2-b^2}{a^2-c^2}}.$$

For the first part see § 340.

For the second part we have, by § 344 (1), to evaluate

$$V = 2\pi\gamma\rho(1-m^2)$$
$$\times \int_0^{\frac{1}{2}\pi} \sin\theta d\theta / \{(\sin^2\theta/a^2+\cos^2\theta/c^2)(\sin^2\theta/b^2+\cos^2\theta/c^2)\}^{\frac{1}{2}},$$

i.e. I, where $2V(1-m^3)abc = 3M(1-m^2)I$.

Put

$c = a\cos\chi$; $\tan^2\theta = (a^2\cos^2\psi - c^2)/c^2\sin^2\psi$; $\kappa^2 = (a^2-b^2)/(a^2-c^2)$;

so that $b^2 = a^2(1-\kappa^2\sin^2\chi)$. It will be found that

$$I\sin\chi = abc\int_0^{\chi} d\psi/\sqrt{1-\kappa^2\sin^2\psi}.$$

468. AB is a uniform straight line of length a, the particles of which attract according to the law of the inverse square. PQR is a circular arc, of radius a and in the same plane with AB, and capable of turning about its centre B. BP, BQ, BR make angles 2α, 2θ, 2β with AB produced. Prove (1) that the potential of AB at Q is $\log(1+\sec\theta)$, (2) that, in order that the arc may be in equilibrium, a couple about B of moment varying as $\log\{(1+\sec\alpha)/(1+\sec\beta)\}$ must be impressed upon it.

(Prove, and use, the fact that, if a body, which is movable about a fixed axis, is in the presence of an attracting mass of which the potential is V, then $\iiint dm\, dV/d\omega$ is the moment of the couple about the axis required to maintain equilibrium, ω being the angle between a fixed plane and one passing through the axis and through the element dm.)

469. Prove that the potential of a circular disk of uniform density and radius a at a point in its plane distant c from its centre ($c<a$) is

$$\frac{4M\gamma}{\pi a}\int_0^{\frac{\pi}{2}}(1-a^{-2}c^2\sin^2\theta)^{\frac{1}{2}}d\theta.$$

Take the point as the origin of polar co-ordinates.

470. Show that the potential of a thin elliptic homoeoid, of unit density, at an external point $x'y'z'$ is

$$V = 2\pi\gamma\, abc \times \frac{t}{p}\int_{\lambda}^{\infty} \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}},$$

a, b, and c being the semi-axes of the homoeoid, p the perpendicular on the tangent plane and t the thickness at any point of the homoeoid, and λ the positive root of the equation

$$\frac{x'^2}{a^2+\lambda}+\frac{y'^2}{b^2+\lambda}+\frac{z'^2}{c^2+\lambda}=1.$$

A homoeoid is a shell bounded by two similar and similarly situated surfaces; e.g. the shell in Ex. 467.

471. The potential of a homogeneous wire in the form of any plane oval curve without inflexions at a point inside is equal to the potential of a homogeneous wire of the same material and thickness whose form is the pedal of the given oval with respect to the point.

472. Find the potential of a solid ellipsoid at an internal point (x, y, z) in the form

$$\frac{3}{4}M\left(I+2x^2\frac{dI}{da}+2y^2\frac{dI}{db}+2z^2\frac{dI}{dc}\right),$$

where $I=\int_0^{\infty}\{(a+u)(b+u)(c+u)\}^{-\frac{1}{2}}du.$ See § 344 (7).

473. If S is a homogeneous solid ellipsoid and C its focal ellipse, prove that it is possible to coat the area bounded by C with matter so that the two masses have the same potential at any external point.

(See Lipschitz, *Crelle's Journal*, Bd. 61, p. 34.)

By § 344 (19) we see that the ratio of the potentials of two confocal ellipsoids at a point external to both is the same as the ratio of their masses. Hence (see Ex. 452 above), if we coat the area enclosed in C with a layer of matter so that the bounding surface, C', is

$$x^2/(a^2-f^2)+y^2/(b^2-f^2)+z^2/(c^2-f^2)=1,$$

where $f\rightarrow c$, and if the mass of this layer is equal to that of S, say M, the two masses have the same potential at any external point. It remains to find the density σ at any point $P(x, y, 0)$ of the area.

The volume of the prism of section $dxdy$ cut off by C' is $2z\,dxdy$. If ρ is the density of C', the mass of the prism $=2z\rho\, dxdy$. Now

$$3M = 4\pi\rho\,\{(a^2-f^2)(b^2-f^2)(c^2-f^2)\}^{\frac{1}{2}}$$

and

$$z=(c^2-f^2)^{\frac{1}{2}}\,\{1-x^2/(a^2-f^2)-y^2/(b^2-f^2)\}^{\frac{1}{2}}.$$

Hence the mass of the prism is

$$3M\,\{1-x^2/(a^2-f^2)-y^2/(b^2-f^2)\}^{\frac{1}{2}}\,dxdy/2\pi\,\{(a^2-f^2)(b^2-f^2)\}^{\frac{1}{2}},$$

and, if the area is coated on both sides, this $=2\sigma\, dxdy$. Proceeding, then, to the limit $f\rightarrow c$, we have

$$\sigma = 3M\,\{1-x^2/(a^2-c^2)-y^2/(b^2-c^2)\}^{\frac{1}{2}}/4\pi\,\{(a^2-c^2)(b^2-c^2)\}^{\frac{1}{2}}.$$

474. If M is the mass of the homogeneous solid, nearly spherical, ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, prove that the potential at an external point (x, y, z) distant R from its centre is

$$\frac{M}{R}+\frac{Mc^2}{5R^5}\{(2k-k')x^2+(2k'-k)y^2-(k+k')z^2\},$$

where $ck=a-c$, $ck'=b-c$. See § 354.

475. The generating circle of a hollow thin-walled anchor ring is of radius a and has its centre distant c from the axis of the ring. Show that the potential at a point on the axis distant a/κ from the centre of a generating circle is $8\pi\sigma ckE$, where

$$E\equiv\int_0^{\pi/2}(1-\kappa^2\sin^2\phi)^{\frac{1}{2}}d\phi,$$

σ being the surface-density of the matter forming the ring.

Hence, or otherwise, show that if the ring is solid and of uniform density ρ, the potential at the same point is

$$\tfrac{8}{3}\pi\rho ca\{(\kappa^{-1}+\kappa)E-(\kappa^{-1}-\kappa)F\},$$

where $F\equiv\int_0^{\pi/2}(1-\kappa^2\sin^2\phi)^{-\frac{1}{2}}d\phi.$

The following Papers are important in connexion with the potential of an anchor ring: Dyson, *On the Potential of an Anchor Ring* (Phil. Trans., 1893, A, p. 43-); Dixon, *Note on the Potential of Rings* (Proc. of the London Math. Soc., xxviii, p. 439); Hobson, *On some general Formulae for the Potentials of Ellipsoids, Shells, and Discs* (Proc. L. M. S., xxvii, p. 519).

The essential feature of the geometry of Dyson's method is that he describes consecutive spheres about the point, P, at which the potential is to be found. The necessary figure can be constructed as follows:—

Let there be in the plane of the paper the axis Oz and a circle with centre C and radius a; OC, $=c\ (>a)$, being perpendicular to Oz and intersected by the circle in A. Two circles, with centre P and radii ρ and $\rho+d\rho$ respectively, cut the former circle in Q, R; Q', R'. Perpendiculars QL and $RM\ (<QL)$ are let fall on Oz. Let $PC=R$, $\angle PCQ=\psi$, $\angle OCP=\alpha$. Now let the figure in the plane of the paper be supposed to revolve round Oz. The potential at P of the bands traced out by QQ' and $RR'=2\pi\sigma a\,d\psi(QL+RM)/\rho$. Clearly $QL=c-a\cos(\alpha+\psi)$, $RM=c-a\cos(\alpha-\psi)$. We have thus to integrate $4\pi\sigma a/(c-a\cos\alpha\cos\psi)d\psi/\rho$ from $\psi=0$ to $\psi=\pi$.

In the present case $R=a/\kappa$, $c=R\cos\alpha$. Hence

$$V=4\pi\sigma\kappa c\int(1-\kappa\cos\psi)\,d\psi/\sqrt{1+\kappa^2-2\kappa\cos\psi}$$

from $\psi=0$ to $\psi=\pi$. Put $\sin(\phi-\psi)=\kappa\sin\phi$, then

$$d\psi/\sqrt{1+\kappa^2-2\kappa\cos\psi}=d\phi/\sqrt{1-\kappa^2\sin^2\phi},$$

and $\cos\psi=\kappa\sin^2\phi+\cos\phi\sqrt{1-\kappa^2\sin^2\phi}$. Hence V consists of two parts, one of which gives the required result while the other vanishes.

The potential of the solid anchor ring may be determined by a somewhat similar method. The volume formed by the revolution of $QRR'Q'$ is $2\pi\rho d\rho . LM$, being the portion of a sphere cut off by two parallel planes. Since $\rho^2 = R^2+a^2-2aR\cos\psi$, we can obtain for the potential the value $V=2(M/\pi)\int\sin^2\psi d\psi/\sqrt{R^2-2aR\cos\psi+a^2}$ from $\psi=0$ to $\psi=\pi$, or, say, $2MI/\pi$. Since $R=a/\kappa$, we can write

$$I=(2/3R)\int d\psi/\sqrt{1+\kappa^2-2\kappa\cos\psi}$$
$$-\{(\kappa+1/\kappa)/3R\}\int\cos\psi d\psi/\sqrt{1+\kappa^2-2\kappa\cos\psi}.$$

By the transformation used above, we can show that the first of these integrals $=2F$ and the second $=2(F-E)/\kappa$: whence the answer.

(Green's Paper *On the Determination of the Exterior and Interior Attractions of Ellipsoids of variable densities* (Trans. Camb. Phil. Soc., 1835) may also be consulted with profit. It will be found in the volume of his Mathematical Papers edited by N. M. Ferrers in 1871.)

476. Prove that the potential at any point on the axis of z of an anchor ring, whose density at any point varies inversely as ρ, and the equation to whose surface in cylindrical co-ordinates is $(\rho-c)^2+z^2=a^2$, is

$$\frac{4M}{\pi(c^2+z^2)^{\frac{1}{2}}}\int_0^K \operatorname{cn}^2 u\,du,$$

where $k^2=a^2/(c^2+z^2)$, and show that at any external point the potential is given by

$$\frac{M}{\pi}\left\{1+\sum_{n=1}^{n=\infty}(-1)^n\frac{a^{2n}}{n+1}\frac{\lfloor 2n}{(\lfloor n)^3 2^{3n}}\left(\frac{d}{cdc}\right)^n\right\}\int_0^\pi\frac{d\phi}{(z^2+\rho^2+c^2-2c\rho\cos\phi)^{\frac{1}{2}}}.$$

(See the references prefixed to the notes on Ex. 475.)

The general character of a transformation utilized by Prof. Hobson will be more apparent if we consider a ring of elliptic section with equation $(\rho-c)^2/a^2+z^2/b^2=1$. It will be convenient to write R for $\rho-c$. Since the density is a function of ρ only, we can take $2\pi\rho\,d\rho\,dz$ as an element of volume, its mass being $2\pi\mu\,dR\,dz$. The potential at the point $(0, 0, z')$ is $V=2\pi\mu\iint dR\,dz/\{(R+c)^2+(z-z')^2\}^{\frac{1}{2}}$.

The treatment of this integral is facilitated by the use of a discontinuous factor. The nature and object of such a factor can be inferred from an elementary example.

If P is a function of x and y which is equal to unity or to zero according as $1-x^2/a^2-y^2/b^2$ is positive or negative, then

$$\iint F(x,y)\,dx\,dy$$

taken over the interior of the ellipse $x^2/a^2+y^2/b^2=1$ is equal to $\iint PF(x,y)\,dx\,dy$ taken over the whole of the plane xy.

Now it can be shown (see the Note appended to this solution) that $\int_{-\infty}^{\infty}\Gamma(\lambda)e^{n(q+is)}ds/2\pi(q+is)^\lambda n^{\lambda-1}$ is equal to unity if $n>0$, and to zero if $n<0$. Hence, if $n\equiv 1-R^2/a^2-z^2/b^2$, this integral with λ put $=1$ is an appropriate factor; and

$$V/\mu=\iiint dR\,dz\,ds\,e^{n(q+is)}/(q+is)\{(R+c)^2+(z-z')^2+h^2\}^{\frac{1}{2}},$$

the limits of R, z, s being $-\infty$ and $+\infty$, and h being put $=0$ at a later stage. It is found to be advantageous to use the known result $\int_0^\infty e^{-kt}dt/t^{\frac{1}{2}} = \Gamma(\frac{1}{2})\,k^{-\frac{1}{2}}$, and to put $k=(R+c)^2+(z-z')^2+h^2$. This gives

$$V\sqrt{\pi}/\mu = \iiint dR\,dz\,ds\int_0^\infty \text{exp.}\,\{(q+is)-X-Y-th^2\}\,dt/t^{\frac{1}{2}}(q+is),$$

$$\text{where } X = R^2\{(q+is)/a^2+t\}+2Rct+c^2t,$$

$$Y = z^2\{(q+is)/b^2+t\}+2zz't+z'^2t.$$

We shall now invert the order of the integrations, taking R and z first.

Now $\int_{-\infty}^{\infty}\text{exp.}\,(-\alpha R^2-2\beta R-\gamma)\,dR = \sqrt{\pi/\alpha}\;\text{exp.}\,\{(\beta^2-\gamma\alpha)/\alpha\}$.

Hence

$$\int_{-\infty}^{\infty}\text{exp.}(-X)dR = a\sqrt{\pi/(q+is+a^2t)}\;\text{exp.}\,\{-c^2t\,(q+is)/(q+is+a^2t)\}$$

and

$$\int_{-\infty}^{\infty}\text{exp.}(-Y)dz = b\sqrt{\pi/(q+is+b^2t)}\;\text{exp.}\,\{-z'^2t(q+is)/(q+is+b^2t)\}.$$

Accordingly

$$V/\mu ab\sqrt{\pi} = \iint \text{exp.}\,Z\,.\,ds\,dt/\sqrt{t\,(q+is+a^2t)(q+is+b^2t)}\,(q+is),$$

the limits of s being $-\infty$ and $+\infty$ and those of t zero and ∞, while

$$Z \equiv (q+is)\,\{1-c^2t\,/\,(q+is+a^2t)-z'^2t\,(q+is+b^2t)-th^2\}.$$

Now put $(q+is)/\theta$ for t. Then Z becomes

$$(q+is)\,\{1-c^2/(a^2+\theta)-z'^2/(b^2+\theta)-h^2/\theta\} = (q+is)\,U \text{ say,}$$

and

$$V/\mu ab\sqrt{\pi} = \int_0^\infty d\theta\int_{-\infty}^{+\infty} ds\,\text{exp.}\,\{(q+is)\,U\}/(q+is)^{\frac{3}{2}}\{(a^2+\theta)(b^2+\theta)\theta\}^{\frac{1}{2}}.$$

But, if we put $\lambda=\frac{3}{2}$ in the formula quoted above, we see that

$$\int_{-\infty}^{\infty} ds\;\text{exp.}\,\{(q+is)\,U\}/(q+is)^{\frac{3}{2}} = 2\pi U^{\frac{1}{2}}/\Gamma(\tfrac{3}{2}) = 4\sqrt{\pi}\,U^{\frac{1}{2}}$$

or zero according as $U>$ or <0.

$$\therefore\; V/4\mu ab\pi = \int_\phi^\infty d\theta\left\{1-\frac{c^2}{a^2+\theta}-\frac{z'^2}{b^2+\theta}-\frac{h^2}{\theta}\right\}^{\frac{1}{2}}\Big/\left\{(a^2+\theta)\,(b^2+\theta)\,\theta\right\}^{\frac{1}{2}},$$

where ϕ is the positive root of the equation $U=0$.

Now put $h=0$, $b=a$, and we get $\phi=c^2+z'^2-a^2$. To effect the reduction let $a^2+\theta=a^2/k^2y^2$, and $z'=z$ where $k^2(c^2+z^2)=a^2$. Then

$$V/8\pi ak\mu = \int_0^1 dy\sqrt{1-y^2}/\sqrt{1-k^2y^2}.$$

If $y=\text{sn}\,u$, this integral becomes $\int_0^K \text{cn}^2u\,du$.

We have now to determine M. This is $\int\int 2\pi r dr.(\mu/r)dz$, the limits of r being $c \pm \sqrt{a^2-z^2}$, and those of z being $\pm a$. Hence

$$M = 2\pi\mu\int_{-a}^{a} 2\sqrt{a^2-z^2}dz = 2\pi^2\mu a^2,$$

and V has the value given in the question.

In order to find the potential at any external point, we may proceed as follows:—

Put $(z^2+\rho^2+c^2-2c\rho\cos\phi)^{\frac{1}{2}} = X'$ and $(z^2+c^2)^{\frac{1}{2}} = X$; then we can verify that

$$\int_0^{\pi} d\phi/X',$$

and therefore the result of operating upon this integral any number of times with (d/cdc), is a solution of the differential equation $\nabla^2 u = 0$. Hence the expression given in the question satisfies this differential equation; and it will be the potential required, provided that it vanishes at infinity and that at every point on the axis of revolution it is equal to the expression previously obtained for the potential at that point (§ 347).

Put then $\rho = 0$ and observe that

$$(d/cdc)^n\int_0^{\pi} d\phi/X = (-1)^n\pi \,.\, 1 \,.\, 3 \,.\, 5 \ldots (2n-1)/X^{2n+1}.$$

We thus find that

$$\frac{M}{\pi}\left\{1+\Sigma(-1)^n\frac{a^{2n}}{n+1}\frac{\lfloor 2n}{(\lfloor n)^3 2^{3n}}\left(\frac{d}{cdc}\right)^n\right\}\int_0^{\pi}\frac{d\phi}{X}$$

$$= \frac{2M}{Z}\left\{\frac{1}{2}+\Sigma\frac{a^{2n}}{2n+2}\,.\,\frac{1^2\,.\,3^2\ldots(2n-1)^2}{2^2\,.\,4^2\ldots(2n)^2}\frac{1}{X^{2n}}\right\}$$

$$= \frac{4M}{\pi(c^2+z^2)^{\frac{1}{2}}}\left\{\frac{\pi}{4}+\frac{\pi}{2}\Sigma k^{2n}\frac{1^2\,.\,3^2\ldots(2n-1)^2}{2^2\,.\,4^2\ldots(2n)^2(2n+2)}\right\}$$

Also $\displaystyle\int_0^{K}\mathrm{cn}^2 u\,du = \int_0^{\frac{\pi}{2}}\cos^2\theta d\theta/\sqrt{1-k^2\sin^2\theta}$

$$= \frac{\pi}{4}+\Sigma k^{2n}\frac{1\,.\,3\ldots(2n-1)}{2\,.\,4\ldots 2n}\int_0^{\frac{\pi}{2}}\cos^2\theta\sin^{2n}\theta\,d\theta$$

$$= \frac{\pi}{4}+\frac{\pi}{2}\Sigma k^{2n}\frac{1^2\,.\,3^2\ldots(2n-1)^2}{2^2\,.\,4^2\ldots(2n)^2(2n+2)};$$

whence the result.

Note on a definite integral.

As the evaluation of the integral $\displaystyle\int_{-\infty}^{\infty} e^{n(q+is)}ds/(q+is)^m$ presents some difficulties, a sketch of the process may usefully be given here. For brevity write Q for $q+is$.

Consider $I = \displaystyle\int_{-\gamma}^{\gamma} e^{nQ}ds/Q^m$, and assume m, n, q to be positive

quantities. Put $nQ = z$; then $I = (n^{m-1}/i)\int dz e^z/z^m$ between the limits $n(q-\gamma i) = N'$, say, and $n(q+\gamma i) = N$, say. We shall utilize the theorem $\Gamma(m-1)/z^{m-1} = \int_0^\infty e^{-tz}t^{m-2}dt$ to introduce a second variable t, and we shall divide up the range 0 to ∞ into four intervals (i) 0 to $1-\tau$, (ii) $1-\tau$ to $1+\tau$, (iii) $1+\tau$ to h, (iv) h to ∞, where $h\tau = 1$ and $\tau \longrightarrow 0$. Now, since $z = n(q+is)$ and $nq > 0$, $\therefore$ e^{-tz} is numerically < 1, and the numerical value of $\int e^{-tz}t^{m-2}dt$ for interval (ii) $< \int t^{m-2}dt$ for the same interval, i.e.

$$< \{(1+\tau)^{m-1} - (1-\tau)^{m-1}\}/(m-1) < \lambda\tau,$$

where λ is a finite constant. This portion of the integral accordingly $\longrightarrow 0$ as $\tau \longrightarrow 0$. Now take the integral for (iv) and write $tz = x$. The integral becomes $z^{-(m-1)}\int_{hz}^\infty e^{-x}x^{m-2}dx$, which, by repeated integration by parts,

$$= z^{-(m-1)}e^{-hz}\{(hz)^{m-2} + (m-2)(hz)^{m-3} + (m-2)(m-3)(hz)^{m-4} + \ldots\},$$

and this, if $hz > m-1$,

$$< z^{-(m-1)}e^{-hz}(hz)^{m-2}(m-1) < e^{-hz}h^{m-1} \longrightarrow 0 \text{ as } \tau \longrightarrow 0.$$

But, when we introduce these partial evaluations into the integral I, they are multiplied by $\int dz e^z/z$ taken between the limits N' and N. The numerical value of this factor = that of $e^{nq}\int_{-\gamma}^{\gamma} e^{nis}ds/(q+is)$, which is $< e^{nq}\int_{-\gamma}^{\gamma} ds/(q^2+s^2)$, as can be seen in one way by replacing the exponential function by sine and cosine and then rationalizing the denominator. This factor is $\therefore$ $< \pi e^{nq}/q$, which is finite.

Hence $I = \{n^{m-1}/i\,\Gamma(m-1)\}\int_0^h t^{m-2}dt\int_{N'}^{N} dz e^{z(1-t)}/z$, the interval from $t = 1-\tau$ to $t = 1+\tau$ (which $\longrightarrow 0$) being omitted.

We cannot put $\gamma = \infty$ without investigation. We have $\therefore$ to consider the limit of the expression $n^{m-1}\int_0^h t^{m-2}dt\int_N^{N_1} dz e^{z(1-t)}/z$, where $N_1 = nq + \infty i$. The inner part of this $= \int_\gamma^\infty ds\, e^{FQ}/Q$, if $F \equiv n(1-t)$ for brevity, and this integral $= e^{Fq}\int_\gamma^\infty ds(\cos Fs + i\sin Fs)(q-is)/(q^2+s^2)$.

We are thus concerned with four integrals. The integrands $(q\cos Fs)/(q^2+s^2)$, $(q\sin Fs)/(q^2+s^2)$ are increased, if we substitute unity for $\cos Fs$ and $\sin Fs$, and thus the corresponding integrals are less than quantities which are proportional to $1/\gamma$ and $\therefore \longrightarrow 0$. In the other two the quantity $s/(q^2+s^2)$ is positive and continually decreasing. Hence, by the Second Theorem of the Mean, the integral $\longrightarrow \mu e^{Fq}.\gamma/F(q^2+\gamma^2)$, where μ is finite, and this expression $\longrightarrow \mu e^{Fq}/F\gamma$

and $\therefore \to 0$ as $\gamma \to \infty$. In this manner we can satisfy ourselves that the integral from γ to $\infty \to 0$, and the corresponding integral from $-\infty$ to $-\gamma \to 0$. Hence, when $\gamma \to \infty$,

$$\int_{-\gamma}^{\gamma} e^{nQ} ds/Q^m = \{n^{m-1}/i\Gamma(m-1)\} \int_0^h t^{m-2}dt \int_{-\infty}^{\infty} e^{z(1-t)}dz/z,$$

where, as before, the portion from $t = 1-\tau$ to $t = 1+\tau$ is omitted.

Now, by a Theorem of Cauchy (see Forsyth, *Theory of Functions*, chap. ii, or Whittaker, *Modern Analysis*, chap. iii) the integral $\int e^{b\zeta} d\zeta/(\zeta - p)$ (where $\zeta \equiv x + yi$), taken round any contour, $= 2\pi i e^{bp}$ if the point $\zeta = p$ lies within the contour, and $= 0$ if that point lies without it. Here $p = 0$, i.e. the critical point is the origin, and it will be found that the sign of b is of importance.

Let u, v, w be three positive quantities, the last of which is ultimately to be increased without limit. Take as the contour of integration the rectangle $ABCD$, the co-ordinates of the vertices being $A\,(u, -w)$, $B\,(u, w)$, $C\,(-v, w)$, $D\,(-v, -w)$. Then the integral of the function $(\exp.\ \zeta)\,d\zeta/\zeta$ taken round this contour is the sum of the four integrals of the same function taken in succession along AB, BC, CD, DA. Now along AB x has the constant value u: hence for that segment the integral $= \int_{-w}^{w} \exp.\ (u+yi)\, d \log (u+yi)$, which, when $w \to \infty$, we will denote by $F(u)$. Along BC, $y = w$, and the integral

$$= \int_u^{-v} \{\exp.\ (x+wi)\}\,(x-wi)\,dx/(x^2+w^2)$$

$$= \int_u^{-v} e^x \{x \cos w + w \sin w + i\,(x \sin w - w \cos w)\}\, dx/(x^2+w^2),$$

which $\to 0$ as $w \to \infty$. Along CD the integral $= -F(-v)$, and along DA it $\to 0$ as $w \to \infty$. Accordingly, since the origin lies within the contour $ABCD$, we have, by Cauchy's Theorem, $2\pi i = F(u) - F(-v)$. But, as $v \to \infty$, $F(-v)$ has as a factor the exponential $\exp.(-v)$, which $\to 0$. Hence, if $k>0$, $F(k) = 2\pi i$; if $k<0$, $F(k) = 0$.

$$\text{Now } F(k) \equiv \int_{-\infty}^{\infty} e^{k+yi}\,\{d(k+yi)\}/(k+yi) = i\int_{-\infty}^{\infty} e^{k+yi}dy/(k+yi).$$

In this put $k = bnq$, $y = bns$, and, as before, $n(q+is) = z$: then we see that $F(k) = \int_{-\infty}^{\infty} e^{bz}dz/z$; and, since n, q are positive quantities, $\therefore k$ is of the same sign as b, and so $\int_{-\infty}^{\infty} e^{bz}dz/z = 2\pi i$, if $b>0$, and $= 0$, if $b<0$. Accordingly $\int_{-\infty}^{\infty} e^{z(1-t)}dz/z = 0$ for the interval (iii) and $= 2\pi i$ for the interval (i), and thus $\int_{-\gamma}^{\gamma} e^{nQ}ds/Q^m \to 2\pi n^{m-1}/\Gamma(m)$ on reduction.

By a similar procedure we can prove that $\int_{-\gamma}^{\gamma} e^{-nQ}ds/Q^m \to 0$.

477. The density at any point of a solid sphere of radius a is ρxyz, the centre of the sphere being the origin of co-ordinates; prove that the potential at any external point (x, y, z) is

$$4\pi\rho a^9 xyz/63(x^2+y^2+z^2)^{\frac{7}{2}}.$$

Compare § 354, Ex. 15, 16. Expand xyz/r^3 in a series of Harmonics and find the potential of a shell whose radii are h and $h+dh$. Then integrate from $h=0$ to $h=a$. *Aliter* deduce from Ex. 478.

478. Establish the fact that

$$\int_u^\infty xyzU d\lambda/\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{\frac{3}{2}},$$

where u is the positive root of the equation

$$U \equiv 1-x^2/(a^2+\lambda)-y^2/(b^2+\lambda)-z^2/(c^2+\lambda)=0,$$

is the potential at an external point of the ellipsoid $\lambda=0$ filled with matter the density of which at any point is proportional to xyz; and find an expression for the potential at an internal point.

479. The density at any point of a sphere varies as the square of the cosine of the angle between the line joining the centre to the point and a fixed line. Find the potential at any external point.

480. The density at any point of a sphere, of radius a and mass M, varies inversely as the distance of the point from an external point I, whose distance from the centre of the sphere is f; prove that the potential of the sphere at an internal point, whose distance from the centre is r and whose angular distance from the radius through I is θ, is

$$\frac{3M}{2a^3}\sum_{n=0}^{n=\infty}\left[\left(\frac{a^2}{2n+1}-\frac{r^2}{2n+3}\right)P_n(\cos\theta)\left(\frac{r}{f}\right)^n\right],$$

where $P_n(\cos\theta)$ is the Legendre coefficient of order n.

Show also that this potential may be expressed in the form

$$\frac{3Mf}{4a^3}\int_0^r \frac{(a^2-rx)dx}{\{rx(f^2-2fx\cos\theta+x^2)\}^{\frac{1}{2}}}.$$

481. If V is the potential at any point of an equipotential surface, and if R and R' are the principal radii of curvature at a point P on the surface, and if dn is an element of the normal at P measured outwards, then

$$\frac{d^2V}{dn^2}+\left(\frac{1}{R}+\frac{1}{R'}\right)\frac{dV}{dn}=0.$$

Consider a small element of area $d\omega$ at P. Let the tube of force of which this is a section cut the consecutive level surface in Q, where the section is $d\omega'$. Then the principal radii of curvature at Q are $R+dn$, $R'+dn$, for the element of a line of force between two successive level surfaces is in the limit perpendicular to both. Now

$d\omega'/d\omega = (R+dn)(R'+dn)/RR' = 1+(1/R+1/R')dn$ in the limit; and, the attractions at P and Q being F and $F+(dF/dn)\,dn$ and $Fd\omega$ being equal to $\{F+(dF/dn)\,dn\}\,d\omega'$, we have

$$d\omega'/d\omega = 1-F^{-1}(dF/dn)\,dn.$$

Whence the result.

482. Prove that, if for a body which is symmetrical, both as to shape and to density, about an axis we know a potential function which at all points on the axis outside the body is the potential of the body at these points, this function is the potential at every point outside the body. (See § 347.)

The body is a frustum, of height $2c$ and mass M, of a homogeneous right circular cylinder of radius b. Let $b^2+c^2 \equiv a^2$ and let $c/a \equiv \lambda$. Prove that the potential at the point whose co-ordinates referred to the centre of the solid and its axis are r, μ, is, provided $r>a$,

$$2\frac{M}{c}\sum_{n=0}^{n=\infty}\frac{P_{2n}(\mu)}{(2n+1)(2n+2)(2n+3)}\left(\frac{a}{r}\right)^{2n+1}\frac{d}{d\lambda}P_{2n+2}(\lambda),$$

where P_n denotes the zonal harmonic of order n.

483. The density at a point of a heterogeneous sphere of radius a is expressed by the formula $\alpha+\beta\cos\theta$, where α and β are constants, and θ denotes the angle which a line drawn from the centre of the sphere to the point makes with a fixed axis; prove that the potentials at internal and external points differ from the values which they would have, if the density were uniform and equal to α, by

$$\tfrac{1}{3}\pi\gamma\beta(4ar-3r^2)\cos\theta \text{ and } \tfrac{1}{3}\pi\gamma\beta(a^4/r^2)\cos\theta,$$

where r denotes distance from the centre.

484. If the potential of a distribution of attracting matter is zero for all points outside the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, and equal to $1-x^2/a^2-y^2/b^2-z^2/c^2$ for all points inside, find the distribution of matter.

Deduce that the potential of two confocal homogeneous solid ellipsoids of equal mass is the same at all points external to both.

Since $\nabla^2 V = -4\pi\rho\gamma$ at any point inside the ellipsoid, we find that ρ is the same everywhere inside the bounding surface and $=(\Sigma a^{-2})/2\pi\gamma$.

Now the normal force just outside is zero, while just inside it is dV/dn, where dn is an element of normal measured in the outward direction. Hence $O = 4\pi\gamma\sigma+dV/dn$ (§ 322), where σ is the superficial density. Also

$dV/dn = \{(\partial V/\partial x)^2+(\partial V/\partial y)^2+(\partial V/\partial z)^2\}^{\frac{1}{2}} = 2\,\{\Sigma\,(x^2/a^4)\}^{\frac{1}{2}} = 2/p$, if p is the perpendicular from the origin on the tangent plane, $\therefore\ \sigma = -1/2\pi p\gamma$. Thus the homogeneous ellipsoid is coated with a layer of matter, the density of which $\propto 1/p$.

If (a, b, c), (a', b', c') are the semi-axes of two concentric, coaxial, and

confocal ellipsoids, $a'^2-a^2=b'^2-b^2=c'^2-c^2$, or $ada=bdb=cdc$, if $a'=a+da$, &c.; and, since $p^2=a^2l^2+b^2m^2+c^2n^2$, where (l, m, n) are the direction cosines of p, $\therefore pdp=\Sigma al^2da=ada\Sigma l^2=ada$, or the thickness of the shell included between the confocals, which $=dp$, varies as $1/p$. Hence, if the shell consists of matter of suitable uniform density, it is equivalent to a coating of superficial density $+1/2\pi p\gamma$.

Now the distribution determined for the first part of the question has zero potential at any external point P. Let V be the potential of the shell at P. Then, by addition, V is the potential at P of (i) the ellipsoid, (ii) the coating of density σ, (iii) the shell taken together, and (ii) and (iii) counterbalance each other. Hence the shell and the ellipsoid have the same external level surfaces. Now see § 344.

485. The density at any point of a sphere of radius a and mass M varies as the square of the distance from a fixed diameter; show that the potential at any external point is

$$\frac{M}{r}+\frac{a^2M}{14r^5}(3\rho^2-2r^2),$$

where r and ρ are the distances of the external point from the centre and the fixed diameter of the sphere.

486. Two bowls are formed, each by hollowing out a hemisphere from a concentric hemisphere. Each bowl is homogeneous, and the densities and dimensions are such that each exerts an equal attraction at the centre of its plane face. The bowls are placed face to face so that the planes of their faces and the centres of those faces are coincident. Prove that the common centre is a triple point on that equipotential surface of the bowls which passes through it.

Show that $\partial V/\partial x=\partial V/\partial y=\partial V/\partial z=(\partial/\partial x)^2\,V=\ldots=\ldots=0$ at the common centre.

487. The potential at any point P due to a homogeneous hemisphere of radius a and mass M is

$$3\gamma M\left\{\frac{1}{3r}+\frac{1}{8}\frac{aP_1}{r^2}+\sum_3^\infty\left(-\tfrac{1}{2}\right)^n\cdot\frac{1\,.\,3\,.\,5\ldots(2n-5)}{n\,!}\cdot\frac{a^{2n-3}P_{2n-3}}{r^{2n-2}}\right\},$$

where r is the distance of P from the centre, and P is in space void of matter on the same side of the base of the hemisphere as the attracting matter.

488. A mass M is distributed over a spherical surface, centre C, so that its density at any point Q is $\rho(k/OQ)^3$, O being an external point, and k the length of the tangent from O. Show that

$$4\pi\rho k\,.\,CQ^2=M\,.\,OC,$$

and the potential at any point P is $M\,.\,OC/(CQ\,.\,OP)$, or $M/O'P$, according as P lies within or without the sphere, O' being the inverse of O with respect to the sphere. (§ 345 and Ex. 1 appended to it.)

489. If V_0 is the potential of a solid sphere of radius a_0 for any law of attraction which depends only on the relative positions of the attracting points, prove that

$$\frac{1}{a}\left(2\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\int_0^a V_0\,a_0\,da_0$$

is the potential of a spherical shell of radius a and of surface density $(3\cos^2\theta-1)$, where θ is the colatitude measured from the z-axis, the point at which the potential is to be found being external to the sphere of radius a.

If the law of force is the inverse n^{th} power of the distance, prove that the formula holds also for internal points if $n < 3$.

490. The sphere $x^2+y^2+z^2 = a^2$ is divided into two parts by the plane $z = 0$, and one hemisphere is filled uniformly with attracting matter. Show that the potential of this hemisphere at any point external to the sphere is given by

$$\frac{M}{r}+\frac{M}{a}\left[\frac{3}{8}\frac{a^2}{r^2}P_1+\ldots+(-1)^n\frac{3.1.1.3.5\ldots\overline{2n-5}}{2^n\,\underline{|n}}\frac{a^{2n-2}}{r^{2n-2}}P_{2n-3}+\ldots\right],$$

where M is the mass of the hemisphere, r the distance of the point from the centre, and P_i is the zonal surface harmonic of degree i.

Find corresponding expressions for the potential at any point inside the other hemisphere, and at any point *on* the sphere.

491. Prove that the potential on itself of a gravitating mass is

$$\iiint R^2\,dx\,dy\,dz/8\pi\gamma,$$

where R is the resultant force at (x, y, z), and the integration extends through all space. Apply this to find the self-potential of a uniform solid sphere. (§ 345 and Ex. 1 appended to it.)

492. The density at any point of a sphere, of radius a and mass M, varies as x^2y^2, where the axes of co-ordinates are three lines at right angles through the centre of the sphere. Prove that the only part of the potential at any external point P (R, θ, ϕ) which depends on the azimuth ϕ of the plane through P and the axis of z is

$$\tfrac{35}{132}Ma^4R^{-5}\sin^4\theta\sin^2 2\phi.$$

493. Show (1) that the gravitation potential of a uniform sphere of radius a and density ρ at an internal point distant r from its centre is $\frac{2}{3}\pi\rho\gamma(3a^2-r^2)$;

(2) that the work done by gravitational forces during a change of configuration of a system of masses is equal to the increase of

$$\tfrac{1}{2}\int V dm,$$

where dm is an element of mass and V the potential of the system at the point where dm is.

Two spheres, each of radius a and density ρ, are placed so as to touch one another. Find the work done by the attractions of the matter composing the spheres if they coalesce so as to form a single sphere of the same density.

The work done in forming a sphere of radius a by bringing all the matter from an infinite distance

$$= \frac{1}{2}\int_0^a \tfrac{2}{3}\pi\rho\,(3a^2 - r^2)\,.\,4\pi\rho r^2 dr = 16\pi^2\rho^2 a^5/15.$$

Accordingly, to form two such spheres and to bring their centres to a distance $2a$ apart, the work to be done

$$= 32\pi^2\rho^2 a^5/15 + \int_{2a}^{\infty} (\tfrac{4}{3}\pi\rho a^3)^2 dr/r^2 = 136\pi^2\rho^2 a^5/45.$$

To form a single sphere of radius $(2a^3)^{\frac{1}{3}}$ the work to be done

$$= 16\,.\,2^{5/3}\pi^2\rho^2 a^5/15.$$

The answer is therefore $8\pi^2\rho^2 a^5 (4\sqrt[3]{2} - 17/3)/15$.

494. A self-attracting system (for the law of the inverse square) is made up of two parts, M and M'. Prove that to remove M' and scatter its particles beyond the region of all force requires an amount of work equal to

$$\int V dm' + \tfrac{1}{2}\int V' dm',$$

where V and V' are the potentials at the position of any element, dm', of M' due, respectively, to M and M'.

From a homogeneous sphere of radius a centimetres is removed and scattered a spherical portion of radius b cm., the distance between the centres of the whole and the removed sphere being c cm. Prove that, if the mass removed is m grams, the work expended is

$$\tfrac{3}{2}\gamma m^2 (a^2 - \tfrac{3}{5}b^2 - \tfrac{1}{3}c^2)/b^3 \text{ ergs}.$$

495. Prove that the work done by the mutual attractions of the parts of a thin uniform circular disk as they come together from a state of infinite diffusion is $8\gamma m^2/3\pi a$, where a and m denote the radius and mass of the disk.

496. Two equal liquid spheres, at a great distance apart, have been formed out of material which originally was widely dispersed; and they subsequently come together and unite in one sphere. Show that the amount of gravitational energy which runs down in the second process is approximately 0·587 of that which ran down in the first.

497. For what laws of attraction, which are functions of the distance only, is the resultant attraction of a spherical shell on an external particle the same as if the matter of this shell were collected at its centre?

This question is treated by Laplace in his *Mécanique céleste.*

Suppose that r and $r + dr$ are the radii and C the centre of the bounding spheres, ρ the density of the matter, P the external point,

$c = CP$; Q any point of the shell, $f = QP$. Then, if $\phi(f)$ is attraction exerted at P by unit mass at Q, the resultant attraction due to the shell will be in the direction PC and will be

$$= \pi\rho r dr c^{-2} \int_{c-r}^{c+r} (f^2 + c^2 - r^2)\,\phi(f)\,df.$$

Now let $\phi(f) = d\phi_1(f)/df$, $f\phi_1(f) = d\psi(f)\,df$. Then we can prove that this resultant attraction

$$= 2\pi\rho r dr d\{c^{-1}\psi(c+r) - c^{-1}\psi(c-r)\}/dc,$$

and by the question this is to be $= 4\pi\rho r^2 dr \phi(c)$; and, if

$$\psi(c+r) - \psi(c-r) \equiv R,$$

we find, by integration, that $R = 2cr\int\phi(c)\,dc + Uc$, where U may contain r, since we have integrated with respect to c. From the nature of R we see that $\partial^2 R/\partial r^2 = \partial^2 R/\partial c^2$; and $\partial^2 R/\partial r^2 = c d^2 U/dr^2$, while $\partial^2 R/\partial c^2 = 2r\phi(c) + 2cr\phi'(c)$. From this we deduce that $2\phi(c)/c + \phi'(c)$, which does not contain r, is equal to $(1/2r)\,d^2U/dr^2$, which does not contain c. Therefore each expression is equal to a constant, $3A$, say; whence $\phi(c) = Ac + B/c^2$, where B is another constant. Accordingly the only laws are those of the direct distance, the inverse square, and one compounded of the two.

In a similar manner it can be shown that the law of the inverse square is the only one for which the shell attracts an internal particle equally in all directions.

It is important to note, and it can easily be shown, that, if the particles of *any* body attract with a force varying as the direct distance, the resultant attraction of the body is the same as if the whole mass of the body were collected at its C.G.

498. Prove that, if V_1 is a function, of which the first and second differential coefficients with respect to the co-ordinates are continuous within a closed surface S, and if V_0 is a function, which has the like properties outside S and tends to zero at infinite distances in the order r^{-1} at least, if, further, V_1 and V_0 satisfy Laplace's equation in their respective regions, then both are expressed by the formula

$$\frac{1}{4\pi}\iint (V_0 - V_1)\frac{dr^{-1}}{d\nu_0}\,dS - \frac{1}{4\pi}\iint \frac{1}{r}\left(\frac{dV_0}{d\nu_0} + \frac{dV_1}{d\nu_1}\right) dS,$$

where $d\nu_0$ and $d\nu_1$ are the elements of the normal to the surface-element dS drawn respectively outwards and inwards, and r is the distance of this element from the point at which the function is estimated.

499. Show that if two distinct distributions of matter have the same potential at every point of a finite region, they will have the same potential at any point of space which can be reached by a continuous path beginning at the region and not passing through any point where matter belonging to either system is situated.

500. A finite system of gravitating matter lies wholly within a closed surface. Show that, if its potential at every point of this surface has an assigned value, its potential at any point outside the surface is single-valued.

If the closed surface is an infinite cylinder such that the product of the perpendicular distances at any point on it from n fixed parallel straight lines within it is constant, and is also an equipotential surface for the system, show that the external equipotential surfaces satisfy a similar condition.

Also, if the parallel lines cut a plane perpendicular to them at the vertices of a regular polygon, show that the equations of lines of force in this plane may be written in the form

$$r^n \sin(n\theta + \alpha) = a^n \sin \alpha.$$

501. A point O is taken outside a closed surface S, and a function G is determined by the conditions that it is harmonic at all points outside S with the exception of O, is zero on S and becomes infinite at O, in such a way that the product $G \times$ (distance from O) has unity for its limit at O. Prove that the value at O of the function V, which is harmonic at all points external to S, and becomes equal to a given function F on S, is

$$\tfrac{1}{4}\iint \pi^{-1} F dS\, dG/dn,$$

where dn is the element of the normal to S drawn outwards.

Determine G and dG/dn for a sphere.

502. A number of masses of positive and negative sign attracting according to the law of inverse squares are contained within a sphere of radius a. Prove by Green's Theorem or otherwise that, if R is the resultant force-intensity at any point external to the sphere, then

$$\int R^2 dv = -\int V \frac{dV}{dn} dS,$$

where the volume-integral on the left applies to space external to the sphere and the surface integral on the right applies to the sphere itself.

503. Thirteen equal homogeneous spheres are arranged as follows: six of them A, B, C, D, E, F each touch a sphere O, and their centres lie in a plane through the centre of O. Three spheres, P, Q, R are placed on the same side of the above plane so that P touches A, B, and O; Q touches C, D, and O; and R touches E, F, and O. Three other spheres P', Q', R' are placed on the opposite side of the above plane so that

(1) P' touches A, B, and O; Q' touches C, D, and O; and R' touches E, F, and O;

(2) P' touches B, C, and O; Q' touches D, E, and O; and R' touches F, A, and O.

Determine which of the two arrangements has the greater gravitational potential energy.

504. Prove that the potential of a homogeneous ellipsoid, which is such that the differences between the squares of its axes are quantities whose squares may be neglected, is (to this order of approximation) $M/R+(A+B+C-3I)/2R^3$ at *any* external point.

505. From the potential produced at every point by a uniform surface distribution on a sphere, deduce that produced by a distribution on a sphere when the surface-density varies inversely as the cube of the distance from a given point.

*506. If two different masses of equal amounts have the same external level surfaces, prove that $\int u\,dm$ is the same for each body, where dm is an element of mass and u is any function satisfying the differential equation $d^2u/dx^2+d^2u/dy^2+d^2u/dz^2=0$ at all points within a level surface which encloses both bodies.

507. A particle is in equilibrium under the action of three central attractive forces of equal strength, the law of force being inversely as the distance. Show that the particle is at a focus of the ellipse which touches at their middle points the lines joining the centres of force.

508. A variable sphere cuts a uniform thin spherical attracting shell of mass M; prove that the attraction at the centre of the sphere of the portion of the shell outside it is $\gamma M\sin^2\chi/d^2$, where 2χ is the angle at which the spheres cut, and d is the distance between their centres.

509. Prove that the attraction parallel to its axis of a homogeneous solid hemisphere at a point on the circumference of its base is $\frac{4}{3}\gamma\rho a$, where ρ is the density of the hemisphere, a its radius.

510. Prove that the component of attraction parallel to the axis of z at an internal point (x, y, z) of the homogeneous solid of density ρ bounded by the spheroid $(x^2+y^2)(1-e^2)+z^2=a^2$ is

$$4\pi\rho z\{e-\sqrt{1-e^2}\sin^{-1}e\}/e^3.$$

511. Express the component attraction parallel to the largest axis, a, of a solid ellipsoid at a point on the surface in the form

$$3Mxk(a^2-b^2)^{-\frac{3}{2}}\{F(k,\phi)-E(k,\phi)\},$$

where $k^2(a^2-c^2)=(a^2-b^2)$ and $a\cos\phi=c$.

512. If matter is distributed uniformly on a plane circular area of radius a, and if P is a point on a perpendicular to the plane which intersects the circumference of the circle at Q, show that the force at P parallel to the radius through Q is

$$\rho\left[\frac{2y^2+4a^2}{a(y^2+4a^2)^{\frac{1}{2}}}K-\frac{2(y^2+4a^2)^{\frac{1}{2}}}{a}E\right],$$

where ρ is the density per unit area of the surface, y the distance PQ, and K and E the complete elliptic integrals of the first and second kind, the modulus being $2a/\sqrt{y^2+4a^2}$.

513. The attraction at all external points of an anchor ring (whose equation is $[\sqrt{x^2+y^2}-c]^2+z^2=a^2$), filled with matter the density of which at any point varies as $2-c/(x^2+y^2)^{\frac{1}{2}}$, is the same as the attraction of the same amount of matter spread over the surface of the ring in a thin layer of uniform density and thickness. (See Ex. 475.)

514. A uniform plane lamina of areal density σ is bounded by the polar curve $r=f(\theta)$. Assuming that r has only one value for a given value of θ, prove that, at a point on the line through the pole perpendicular to the plane of the lamina, the component of the attraction of the lamina along this line is

$$\gamma\sigma\int\left(1-\frac{c}{\sqrt{c^2+\{f(\theta)\}^2}}\right)d\theta$$

between proper limits, where c is the distance of the point from the pole and γ is the constant of gravitation.

If the lamina is bounded by the two loops of the lemniscate $r^2=c^2\cos 2\theta$, show that its resultant attraction at the point is

$$\gamma\sigma\{\pi-2\sqrt{2}\log(\sqrt{2}+1)\}.$$

515. Show that any two masses attract each other approximately as if the mass of each were collected at its C. G., when the square of the ratio of the greatest linear dimension of either body to their distance apart can be neglected.

516. A semi-elliptic lamina is bounded by the major axis, and each element α of its area is attracted by two equal forces $\mu(r+r')\alpha$, one to each focus of the ellipse, r and r' being its distances from the foci. Prove that the resultant attraction on the lamina is $8\mu(a^3-c^3)/3$, where $2a$ is the major axis and $2c$ the distance between the foci.

517. A solid homogeneous spherical planet of mass M and radius R is covered by a homogeneous sea of mass $3cM/2R$ whose depth varies owing to the attractions of other bodies and is $c(1+\cos\theta)$, where θ is the north polar distance, and c is very small in comparison with R. Compare the densities of the planet and sea, and find to the first order the attraction of the sea only at the North Pole and sea level.

The surface of the sea forms a sphere of radius $R+c$ about a centre distant c from that of the planet. Hence the density of the sea $=$ its mass; its volume $=3cM/2R$; $4\pi\{(R+c)^3-R^3\}/3=$ half that of the nucleus. The force at the sea level at the North Pole

$$=\gamma[\{M+3cM/2R\}/(R+c)^2-M/(R+2c)^2]=7\gamma cM/2R^3.$$

518. From a homogeneous shell bounded by concentric spherical surfaces a portion is cut out by a cone having its vertex at the common centre; prove that the attraction of the rest of the shell upon the portion cut out is $\frac{1}{3}\pi^2\gamma\rho^2\sin^2\alpha\,(a-b)^2(a^2+2ab+3b^2)$, where a and b denote the external and internal radii, ρ the density, and 2α the vertical angle of the cone.

519. Prove that, if the shell in Ex. 518 is very thin and the cone very slender, the attraction is approximately equal to $\frac{1}{2}mR$, where m denotes the mass of the portion cut out and R the attraction of the complete shell at a point on the outer surface.

520. Show that the attraction of a thin homogeneous shell bounded by two similar, coaxial ellipsoids on an external particle is in the direction of the axis of the cone which has its vertex at the particle and envelopes the shell.

See § 342. The axis of the enveloping cone is normal to the confocal ellipsoid through its vertex.

521. Show that a particle cannot in general rest in equilibrium on a solid ellipsoid under the influence of its attraction except at the ends of the principal axes, and that it is in stable equilibrium at the ends of only one of the axes.

522. A solid of uniform density ρ is bounded by the cylinder $x^2+y^2=a^2$ and the planes $z=\pm h$. Prove that the attraction at a point on its axis at a small distance z from the centre is approximately $4\pi\rho z\{1-h/\sqrt{h^2+a^2}\}$, and that the attraction at a point in the plane $z=0$ at a small distance r from the centre is approximately $2\pi\rho hr/\sqrt{h^2+a^2}$.

523. A mountain range may be assimilated to a portion of an infinite cylinder whose cross-section is a segment of a circle of base $2b$ and height h. Prove that, at a point at the foot of the range, the deviation θ of the plumb-line from the vertical is approximately

$$3\rho b\tan^{-1}(h/b)/\pi\sigma R,$$

where ρ denotes the density of the mountains, σ the mean density of the earth, and R the radius of the earth.

Find the deviation in seconds when $h=2$, $b=10$, $R=4000$, $\rho=3{\cdot}5$, $\sigma=5{\cdot}5$.

524. A long straight uniform range of mountains of height a, of the form of half an elliptic cylinder bounded by a principal axis $2c$ of the cross-section, stands on a horizontal plane, and is of the same density ρ' as the earth's superficial strata; show that close to its base the plumb-line is deflected by it through an angle $3\rho' ac/2\rho R(a+c)$, where ρ is the mean density of the earth and R its radius. Taking ρ to be $5\frac{1}{4}$ and ρ' to be 3, find roughly how much this would amount to for a mountain chain one mile high and four miles broad.

525. Two uniform spherical shells, which cut orthogonally, attract according to the law $1/r^5$; prove that they are in equilibrium under their mutual attraction. (Consider the masses intercepted by an elementary cone having its vertex at the centre of one shell.)

526. Show how to divide the earth, supposed homogeneous and spherical, by a plane perpendicular to its axis into two parts which exert equal forces on a particle at the North Pole.

527. At each end of a diameter BC of a uniform sphere is attached a mass m, and the system is in presence of a gravitating particle of mass m' at A. Prove that there is a couple tending to turn the sphere about an axis through its centre perpendicular to the plane of the triangle ABC of amount

$$\tfrac{1}{2}\gamma mm'(b \sim c)\sin A(1/b^2+1/c^2+1/bc),$$

where a, b, c, A, B, C are the sides and angles of the triangle.

528. Four small smooth rings, each of mass m, are capable of sliding on a wire which is coincident with the perimeter of the ellipse $x^2/a^2+y^2/b^2=1$, and a particle, also of mass m, is fixed at the point (h, k). Each ring is attracted by each of the others and also by the particle according to the law of the direct distance. Prove that, when there is equilibrium, the ordinates of the rings are the roots of the equation

$$(t^2-b^2)\left[t+b^2k/(5a^2-3b^2)\right]^2 = -a^2b^2h^2t^2/(3a^2-5b^2)^2.$$

(The normals at the rings must intersect at the C. G. of the five masses.)

529. Prove that a uniform inextensible cord can rest in the form of a circle, C, under the action of a repulsive force emanating from a point O in its plane, the force at a point P of the cord being $\mu\,.\,OP/t^4$, where t is the length of the tangent drawn from P to the circle, with centre at O, which is orthogonal to C, and deduce (or otherwise show) that, if O is on C, the law is μ/OP^3. (For this and the following examples see Papers by Professor R. Townsend in the *Quarterly Journal of Mathematics*, vols. 12 and 13.)

530. Prove that, if a uniform inextensible cord, A, is in the plane of a uniform circular ring, C, repelling it according to the law of the inverse cube of the distance, a possible form of A is a circle orthogonal to C.

531. Prove that two uniform inextensible circular cords lying in the same plane and repelling each other according to the law of the inverse cube of the distance will, if they intersect orthogonally and are of sufficient strength at the points of intersection, remain in equilibrium.

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532. If the direction and magnitude of the resultant of two forces, the directions of which are inclined at a given obtuse angle, are known, prove that one of the forces is greatest when the direction of the other is perpendicular to the direction of the resultant.

533. Forces proportional to the sides of a polygon act at their middle points, at right angles to them and outwards. Prove that they are in equilibrium.

If the polygon is a quadrilateral and a framework of light rods, show that the framework will not be in equilibrium unless it is inscribable in a circle. In the latter case find the stresses along the rods.

534. A square window sash weighing W lb. slides vertically in grooves. From the two upper corners sash cords are carried over pulleys and carry two counterpoises each weighing $\frac{1}{2}W$ lb. Show in a diagram the forces acting on the sash when one of the sash cords breaks, and find the least coefficient of friction between sash and grooves that will keep the sash from sliding down, all other friction being neglected.

535. A heavy rod, free to turn about its lower end, which is fixed, rests against a fixed rough horizontal rail. Prove that, provided the rod is long enough, the distance between the extreme positions of the point of contact is $2\mu k \sec\phi \tan\phi/\sqrt{1-\mu^2\tan^2\phi}$, where μ is the coefficient of friction, k the depth of the fixed end of the rod below the level of the rail, and ϕ the angle which the plane containing the rod and the rail makes with the vertical. The thickness of the rod and rail are neglected.

536. A cone of semi-vertical angle $\beta\,[>\tan^{-1}(\frac{1}{2}\tan\alpha)]$ has three small knobs attached to the perimeter of the base, forming the vertices of an equilateral triangle. The cone rests on a plane inclined to the horizon at an angle α, the coefficient of friction between the plane and the knobs being μ $(>\tan\alpha)$, and is so placed that two of the knobs are on a horizontal line below the third. Force is then applied to the cone, down the line of greatest slope passing through the upper knob; and the cone is on the point of slipping when the force is equal to twice the limiting friction at that knob. Show that the coefficient of friction is $\tan^{-1}\{3\tan\alpha\tan\beta/(\tan\alpha+\tan\beta)\}$.

537. A homogeneous rough cylinder of elliptic section rests on an inclined plane so that its generators are horizontal. Prove that, if the plane is sufficiently rough to prevent slipping, and the inclination of the plane to the horizontal is small enough, there are two positions of equilibrium, and that in that position in which the major axis makes the smaller angle with the lines of slope of the plane the equilibrium is stable, and in the other it is unstable.

538. Two equal uniform rods, each of length $2a$, are freely jointed together at one end, and placed in a symmetrical position over a smooth horizontal cylinder of radius b. Their other extremities are connected by a light inelastic string of length $4c$, which does not meet the cylinder. Find the ratio of the tension to the weight of a rod in terms of a, b, c.

539. A framework of seven bars of negligible weight is in the form of three triangles BCD, ABD, ECD. Of these BCD is equilateral with BC horizontal, B to the left of C and D above BC; ABD is isosceles with AB equal to BD, and A to the left of B and below it; ECD is isosceles with EC equal to CD, and E to the right of C and below it and at the same level as A. The bars are jointed smoothly at their common extremities, the ends A and E are supported, and equal weights W are hung from the joints B, C, D, the whole system

being in a vertical plane. Prove the following construction for the stresses in the bars:—From a point O draw lines OS, OT, parallel to AB, AD respectively, and in the senses indicated by the order of the letters, and meeting a vertical line in S and T; on ST take a point V so that $SV = 2\,VT$; from V draw VU parallel to DB to meet a horizontal line drawn through O in U; then the stresses in AB, AD, BD, BC are represented by the lines OS, OT, VU, OU on the same scale as the weight W is represented by VS.

Determine which of the stresses are tensions and which are thrusts.

540. Two smooth bars are fixed in a vertical plane so that the angle between them is $\alpha+\beta$, and they make angles α and β respectively (each less than $\frac{1}{2}\pi$) with the vertical drawn downwards through their point of intersection. To the ends of a uniform bar of weight W and length l are attached small rings which can slide one on each of the fixed bars. Prove that in the position of equilibrium the distances a, b from the point of intersection of the fixed bars to the positions of the rings are given by the equations

$$\frac{a-b\cos(\alpha+\beta)}{\cos\alpha}=\frac{b-a\cos(\alpha+\beta)}{\cos\beta}=\frac{l^2}{a\cos\alpha+b\cos\beta}.$$

541. Two uniform rods AB, AC, each of length b $(>a)$ and of weight w, are freely jointed at A and are supported by an elastic string of natural length a which is attached to the joint A and to a point O at a height $a+b$ above a smooth horizontal plane. Show that, if the modulus of elasticity of the string is equal to w, the string will be stretched to twice its natural length in the position of equilibrium.

If the plane is rough $(\mu<1)$ and $b=a$, show that there is only one position of limiting equilibrium; and determine the angle between the rods in terms of the angle of friction.

542. Three forces P, Q, R act in the lines $y=b$, $z=-c$; $z=c$, $x=-a$; $x=a$, $y=-b$, in the positive senses of the axes of co-ordinates. Find the equations of the central axis, and prove that the system is equivalent to a single resultant if $a/P+b/Q+c/R=0$.

543. If a force (X, Y, Z) acts along a generator of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$, and if it is equivalent to an equal force (X, Y, Z) at the origin together with couples L, M, N whose axes are the co-ordinate axes of x, y, z respectively, prove that

$$a^2L^2+b^2M^2=c^2N^2,\ cN=\pm Zab.$$

544. At every point of the positive octant of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ a force μdS acts along the normal, dS being the element of surface. Prove that these forces are equivalent to a single force acting along the line

$$a(x-4a/3\pi)=b(y-4b/3\pi)=c(z-4c/3\pi).$$

(Let the point be x, y, z and p the perpendicular from the origin on the tangent plane. The direction cosines of the normal are $l = px/a$, $m = py/b$, $n = pz/c$. Then for the point

$$X_1 = l\mu dS = \mu dydz\,;\ \therefore\ X = \Sigma X_1 = \mu \iint dydz$$

over the quadrant $= \frac{1}{4}\mu bc$. So

$$L = \Sigma(yZ_1 - zY_1) = \mu\iint ydxdy - \mu\iint zdzdx = \tfrac{1}{3}\mu a(b^2 - c^2).$$

Hence $\Sigma LX = 0$, and Poinsot's Axis is the required line of action.)

545. The line L is the Poinsot's Axis of a system of forces, and C is an elliptic cylinder the generators of which are parallel to L. An additional force P of given magnitude is made to act along a generator of C. Prove that for different lines of action of P the new Poinsot's Axis lies on another elliptic cylinder which is similar and similarly situated to C.

546. Prove that if the resultant of five forces acting along five given straight lines reduces to a single force acting through a given point, its line of action lies in a fixed plane.

547. $ABCD$ and $EFGH$ are two opposite faces of a parallelepiped, of which AE, BF, CG, DH are parallel edges. Show that, if six forces are *completely* represented by the six edges AB, BC, CG, GH, HE, EA, they are equivalent to a couple represented by four times the area of the triangle EGB, and having its axis perpendicular to the plane of this triangle.

548. A heavy string in which the density at any point P is $a(b^2 - s^2)^{-\frac{3}{2}}$, where a and b are constants and s is the arcual distance of P from the lowest point of the string, is suspended from two points in the same horizontal line. Show that it will assume the form of a cycloid if a is properly chosen.

549. Find for a semi-infinite flat plate bounded by a straight edge the resolved attraction normal to the plane of the plate at a point on the normal to this plane drawn through a point on the edge.

550. Prove that in the northern hemisphere a deep crevasse with vertical sides extending a long way east and west increases the apparent latitude of places on its southern edge, and diminishes that of places on its northern edge, by an angle approximately equal (in radian measure) to $3\rho a/4\rho_0 R$, where a is the width of the crevasse, ρ the mean density of rock in the neighbourhood, ρ_0 the mean density of the whole earth, and R its radius.

551. Prove that the potential of a uniform thin ring at any point is

$$4\gamma ma\int_{r_1}^{r_2}\frac{dr}{\{(r^2 - r_1^2)(r_2^2 - r^2)\}^{\frac{1}{2}}},$$

where γ is the constant of gravitation, m the mass per unit of length, a the radius of the ring, r the distance of the point from a point of the ring, r_1 and r_2 the least and greatest values of r. Prove also that the potential may be expressed in the form $8\gamma maK/(r_1 + r_2)$, where

K is the complete elliptic integral of the first kind with modulus $(r_2-r_1)/(r_2+r_1)$.

552. The density at any point within a solid sphere of radius a is $\rho_0+\rho_1 z$, where ρ_0 and ρ_1 are constants, and z is one of the co-ordinates of the point referred to axes having their origin at the centre. Verify the results that the potential at any internal point is

$$\tfrac{2}{3}\pi\gamma\rho_0(3a^2-r^2)+2\pi\gamma\rho_1 z\,(a^2/3-r^2/5),$$

and at any external point is $4\pi\gamma(\rho_0 a^3/3r+\rho_1 za^5/15r^3)$, where r denotes distance from the centre, and γ is the constant of gravitation.

553. Prove that the potential of a uniform circular disk of radius ρ, density 1, and centre C, at a point P, distant x from its plane, can be expressed in the form $2\rho^2\int_\sigma^\infty \sqrt{1-x^2/s-r^2/(\rho^2+s)}\,ds/(\rho^2+s)\sqrt{s}$,

where σ is the positive root of the equation $x^2/\sigma+r^2/(\rho^2+\sigma)=1$, r is the length of the projection of CP on the plane of the disk.

(Heine, *Borchardt's Journal*, Bd. 76, p. 271, investigates this question independently of the potential of an elliptic disk; following him we may proceed thus:—

We premise that, if α is a real positive quantity,

$$\pi/\alpha=\int_{-\infty}^{\infty}dz/(\alpha^2+z^2), \tag{A}$$

and

$$\int_0^{2\pi}d\phi_1/\{a-b\cos(\phi-\phi_1)\}=2\pi/\sqrt{a^2-b^2},\text{ if } a>0,\ a^2>b^2. \tag{B}$$

Now let Q be any point (r_1,ϕ_1) of the disk; then, if the angle between r and r_1 is $\phi-\phi_1$, we have $PQ^2=r^2+r_1{}^2+x^2-2rr_1\cos(\phi-\phi_1)$, and by (A) $\pi/PQ=\int_{-\infty}^{\infty}dz/\{r^2+r_1{}^2+x^2+z^2-2rr_1\cos(\phi-\phi_1)\}$, and, further, by (B) $\int_0^{2\pi}d\phi_1/PQ=2\int_{-\infty}^{\infty}dr/\sqrt{(r^2+r_1{}^2+x^2+z^2)^2-4r^2r_1{}^2}$.

Thus $V=\int_0^\rho r_1dr_1\int_0^{2\pi}d\phi_1/PQ$, or, if $r_1{}^2=u$, $\rho^2=U$, and

$$2R(x^2+z^2)=U+x^2+z^2-r^2$$
$$+\sqrt{U^2+2U(x^2+z^2-r^2)+(x^2+z^2+r^2)^2},$$

$V=\int_{-\infty}^{\infty}\log R\,dz=-2\int_0^\infty zdz\,(d\log R/dz)$, by integration by parts,

i.e. $=-2\int_0^\infty zdR/R$. Introduce a new variable s by means of the equation $(x^2+z^2)/s+r^2/(\rho^2+s)=1$, so that $R=1+\rho^2/s$: we obtain, then, the desired result.)

554. Dividing up an ellipsoid by means of parallel circular sections, use the result of the preceding example to deduce the potential at any external point (Züge, *Mathematische Annalen*, Bd. x).

555. If the particles of the circular disk in Ex. 553 attract according to the law of the inverse fourth power, obtain for the potential at P the expression $\int_\sigma^\infty 2\,ds/3us\sqrt{s+1}$, where

$$u^2 = \rho^2 s\,(s+1) - sr^2 - (s+1)x^2 \text{ (Züge, ibid.)}.$$

556. Let the homogeneous ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, where $a > b > c$, be divided into an infinite number of sections by means of planes parallel to that of x, y. Prove that the potential at an external point (x, y, z) of the section (of axes $2m$, $2n$ and unit density) which is bounded by the planes $z = t$ and $z = t + dt$, is

$$2mndt \int_\rho^\infty F(s)\,ds/s\,(m^2+s)\,(n^2+s),$$

where ρ is the positive root of the equation

$$(t-z)^2/s + x^2/(m^2+s) + y^2/(n^2+s) = 1, \text{ and}$$

$$F(s) \equiv [\{s-(t-z)^2\}\,(m^2+s)\,(n^2+s) - x^2 s\,(n^2+s) - y^2 s\,(m^2+s)]^{\frac{1}{2}};$$

and deduce the potential of the whole ellipsoid.

(Cf. Grube, *Borchardt's Journal*, Bd. 69.)

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